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PULSED LINEAR NETWORKS

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PULSED LINEAR NETWORKS

# PULSED LINEAR NETWORKS

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BY

ERNEST FRANK

*Garden City Research Laboratories,  
Sperry Gyroscope Company, Inc.*

*First Edition*

*Second Impression*

*New York*

*London*

McGRAW-HILL BOOK COMPANY, INC.

1945

PULSED LINEAR NETWORKS

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## PREFACE

The need for understanding and analyzing electrical transients is becoming indispensable in the improvement of existing electrical devices and systems, and especially in the development of many of the forthcoming advances in the electrical field. To fill this need, electrical engineers are finding it necessary to equip themselves with means for recognizing and solving technical problems in transients.

Ultimately, a powerful and general method of transient analysis is the only satisfactory tool, because a method that is not general fails in many instances and sometimes leads to erroneous results. Several fairly general methods of transient analysis exist at present, and they are continually being broadened and improved upon. Because most of them involve mathematics that is often unfamiliar, they are not accessible without the exertion of considerable effort. This is rather unfortunate, but transient analysis is inherently a complicated mathematical subject. Nevertheless, it is possible to proceed to some extent into the subject of transients without introducing new concepts and new methods. Such a penetration cannot be too deep before it becomes clear that a more powerful approach is required.

It is the purpose of this book to analyze transients in mathematical terms that are familiar to most electrical engineers and engineering students in order to set the stage for a more advanced study of the subject. This is done by the exclusive use of the classical method which employs conventional differential equations only. Admittedly, the classical method is limited in its applications; nevertheless, it has utility in many practical instances. Those who study this book should be conscious of its scope and intent. Among the principal things to be gained are

1. A feeling for the distinction between transient and steady-state network behavior.
2. An understanding of the underlying factors governing the transient behavior of networks.

3. A familiarity with the interpretation of mathematical results in terms of the phenomena that they describe.

4. A method of analysis which is applicable to a variety of networks and which does not involve unfamiliar mathematical concepts.

5. A realization of the limitations of the classical method and an appreciation of the need for a more powerful method.

This book deals with fundamentals, and little emphasis is placed upon practical applications except in the last chapter. The results should be useful to both power and communication engineers since many of the transient problems are fundamentally the same in the two related fields. The problems at the end of each chapter are not "practical" problems. They are primarily exercises that illustrate fundamental principles and should help the student to obtain a firmer grasp of the ideas.

It is hoped that undergraduates in their junior or senior year will find that this book offers interesting and useful applications of differential equations to electrical problems. The material presented can be used as a foundation for a one-semester course, can supplement an elementary course in differential equations, or can form the first portion of a comprehensive course in general transient analysis.

Without the constant help and inspiration of the author's wife, Gilda, this book could not have been written, and her efforts are gratefully acknowledged. In addition, thanks are due to Messrs. James E. Shepherd, Walter Selove, and Melvin B. Gottlieb, who offered invaluable suggestions after examining the manuscript. The oscillograms were obtained through the courtesy of the Sperry Gyroscope Company.

ERNEST FRANK.

HEMPSTEAD, N.Y.,  
July, 1945.

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# PULSED LINEAR NETWORKS

## CHAPTER I

### INTRODUCTION

Electrical-network theory is one of the most necessary and one of the most frequently used tools in electrical work. A large portion of this theory is devoted to electrical networks that are in steady operation; *i.e.*, in which voltages and currents are either invariant with time or are varying in a periodic or regular manner. This is well, because the study of the behavior of many networks under steady operating conditions is of great utility. Another classification of network theory is that which applies to networks in which the voltages and currents are variable in a nonperiodic manner; or, in other words, in which an irregular or transient state exists in the network. This behavior is also of interest and has practical significance. In general, if a network changes from one steady operating condition to a different steady operating condition, a transient state exists during the change and becomes nonexistent when the new steady state is reached. Quite often the transient is not desired, but in numerous cases the transient behavior is used to good advantage.

The analyses in this book deal with the transient behavior of networks. It should be recognized from the outset that this is a rather confined subject. The manner in which the network operation undergoes a transitional change depends upon the particular network and upon the causes of the change. Consequently, any theory that attempts to be general must of necessity be exceedingly complex. To avoid this complexity, the theory presented in this book is confined to one particular kind of driving force, a rectangular pulse voltage, which produces the change in network operation.

### TRANSIENTS AND TRANSIENT ANALYSIS

Before discussing the methods by which transient-network operation can be analyzed, it is well to become familiar with

transients and to find a suitable definition of both transient and steady-state operation. A few simple illustrations will serve this purpose.

**1. Examples of Transients.**—The simple act of turning on an electric light with a switch is an exceptionally good example for study. Until the time that the switch is moved, the bulb is dark and emits no light. After the switch is thrown, light appears immediately as far as human perception is concerned. The bulb has passed from one steady state of operation to a new steady state of operation almost instantaneously. In this case, the transient state goes unnoticed and for most practical purposes is unimportant. However, a finite time must elapse between the instant the switch is moved and the instant the light appears, and the fact that human perception fails to identify the transitory state is no assurance that it does not exist. In this small time interval: the switch is moved, the switch mechanism responds and makes an electrical contact, current begins to flow through the bulb filament, and the filament heats up until it emits light. This succession of events takes time and determines the operation during the brief transition period.

An additional transient that is perceptible to the human senses is also started when the light switch is thrown. The bulb is cool, say at room temperature. After the switch is thrown, the bulb begins to get warm and after an appreciable time can become uncomfortably hot. If no air circulates in the room, the bulb will ultimately reach an equilibrium or steady-state temperature that will be different from its temperature before the switch was thrown. Thus a lapse of time is necessary to pass from one steady temperature to a new steady temperature. During this time the bulb temperature is changing in a nonrepeating manner.

Returning to the light emitted by the bulb, which reaches steady intensity almost instantly, assume that the current causing the filament to emit is alternating current. In this case, even though a steady operating condition has been reached, the light is actually pulsating at a regular rate, though the pulsations are not ordinarily perceptible. If the alternating current is of low enough frequency, the pulsations become noticeable, for instance, when the frequency is about 25 cycles per second. Hence, it can be seen that a steady state of opera-

tion is possible even when variations are occurring. The important point is that these variations are regular and recurrent, as distinguished from a transient condition of operation where the variations are nonrepeating.

To carry this example further, suppose the light switch is turned off. The light subsequently goes out almost immediately, although human perception may in this instance lead one to believe that the light remains for an appreciable length of time. This is because the retina of the human eye has certain properties of retention—also a transient or fleeting phenomenon. A finite time is required for the bulb to stop emitting light, *i.e.*, the filament must become cool enough to stop emitting light. Also, a finite and longer time is required for the bulb temperature to recede to room temperature once more. Thus, in turning a light on and then off, there are two distinct transients and two distinct steady-state operating conditions.

Suppose the time between switching the light on and then off is made so short that the bulb never attains its higher equilibrium temperature. In this instance the steady-state temperature during the time that the light is on is never reached. Nevertheless, there is a hypothetical steady-state temperature that may be predicted if enough facts are known.

There is another type of transient that might more aptly be called a disturbance. This type of transient is fundamentally the same as in the case of the electric light, but here both the initial steady-state and final steady-state conditions are the same. For this reason the transient condition usually can be readily distinguished from the steady condition. There are numerous examples of this type of transient. For instance, suppose a man is going to dive into a swimming pool from a springboard. Before he goes out on the board, it is at rest at a certain angle with respect to the water surface. When the man runs to the end of the board, it is displaced. As the man goes into the air, the board vibrates up and down with diminishing amplitude and finally comes to rest in its original position. This vibration is repetitive in a sense, but it is not periodic because of the diminishing amplitude of vibration. It is, therefore, a transient. Another transient of the same type occurs on the water surface. Assume that the water surface is smooth before the man hits. When he hits, a splash occurs

and waves are set up. The surface of the water eventually returns to an undisturbed state after the man has climbed out of the pool.

**2. Characteristics of Transients.**—There are certain significant similarities in both the electric light and the diving examples. These similarities are

1. The steady or prevailing condition is either invariant with time or a regular periodic function of time.

2. The transient or transitional condition is varying with time in a nonrepeating manner and is not periodic.

3. The steady-state condition, which can exist after the transient condition becomes negligible, is not necessarily always attained. In the case of the electric light the switch could be turned off before the bulb became very warm, and in the case of the diving board, other divers could follow the first and keep the board in continual motion, thus forestalling the steady condition of no motion.

4. The steady-state and transient conditions depend upon certain tacit assumptions. In the case of the light bulb it was assumed that the bulb would not burn out, that the line voltage was steady, etc. The diving-board example assumed that the man was not so heavy that he could produce a permanent displacement of the board or that, even worse, he could crack the board in diving. Either of these eventualities would cause the steady conditions after the dive to differ from those before the dive and would also affect the vibration of the board. Similarly, it was assumed that the water level in the pool did not change. All of these assumptions are hidden and for that reason are dangerous. In the electrical cases to be treated the necessary assumptions will be stated beforehand to avoid any pitfalls.

5. In each of the examples given the transient was caused by a sudden change. In the case of the electric light, the switch was suddenly turned on. In the diving example, the man exerted a sudden impact on the springboard. When a transient is produced by this type of disturbance, the *nature* of the transient is governed by factors that are not dependent upon the disturbing force. For instance, in the electric-light case the transient behavior depends upon the characteristics of the light filament, room temperature, etc. In the diving example the

vibration of the board depends upon its means of support, characteristics of the wood, etc. Thus the *nature* of the transient is ruled by factors other than the disturbing force. Nevertheless, this type of disturbing force can affect the *magnitude* of the transient, even though it does not influence its nature. The magnitude of the transient depends upon the line voltage in the electric-light example, while the magnitude of the vibration in the diving example depends upon the initial impact.

In view of the foregoing examples it is possible to define steady-state and transient operation of electrical networks.

*Steady State.*—An electrical network is operating under steady-state conditions when all voltages and currents in the network either are invariant with time or are periodic functions of time.

*Transient State.*—An electrical network is in the transient state when not in the steady state, in other words, when the voltages and currents in the network are variant in a nonrepeating manner.

The distinction between transient and steady-state operation in some networks is not sharp. However, when a single disturbing force is considered, the two states of operation are usually clearly distinguishable.

**3. Transient Analysis.**—There are a number of methods by which networks can be analyzed for their transient behavior. Some of them are

1. Classical method.
2. Heaviside operational calculus.
3. Fourier and LaPlace transforms.
4. Fourier integral.

Because differential equations are generally more familiar than the mathematics required in methods 2, 3, and 4, it is worthwhile to use the classical method wherever possible since it employs differential equations exclusively. In addition, the classical method does not require any electrical concepts beyond the basic electrical laws and is not accompanied by operational or shorthand manipulations that may be difficult for some people to grasp. For these reasons the classical method is used throughout this book.

On the other hand, it is important to realize that the simplicity of this method has some undesirable consequences. Of the four methods the classical is the most limited in its

applications and, except in the case of simple networks, is the most arithmetically laborious. In complex networks the classical method often fails because it is not possible to obtain a sufficient number of "initial" or boundary conditions to evaluate the arbitrary constants that appear in the solution of the differential equation. There is a rather definite degree of network complexity where it becomes vital to use methods other than the classical because of this inherent limitation. However, there are innumerable networks that occur in practice that can be handled conveniently by means of differential equations and in which the arbitrary constants can be evaluated easily. In this book the classical method will be carried to the point beyond which the solution becomes impractically cumbersome or unduly laborious.

### ELECTRICAL NETWORKS

The general question in network transients can be stated as follows: When a disturbing electromotive force that is a function of time is applied to the input terminals of a four-terminal network, what will be the resulting potential difference as a function of time at the two output terminals of the network? As an example, suppose the input is a voltage pulse of some specified shape and amplitude. If this pulse is applied to the two input terminals of a four-terminal network, then what is the shape and amplitude of the pulse that appears at the two output terminals? This question is quite general and the answer cannot be found from ordinary steady-state network theory.

In all methods of transient analysis, certain simplifying assumptions are made in obtaining a solution. This is because the results of an exact analysis, which is usually extremely complicated, are often negligibly different from the results of a simpler approximate analysis.<sup>1</sup> There are essentially two realms of these simplifications: (1) the electrical network and (2) the disturbing force. At the same time, the simplifying assumptions must be close enough to reality so that the results will be approximately applicable to the actual situation. The simplifying assumptions required for the network whose transient characteristics are desired are much the same for all types of

<sup>1</sup> See Chap. IX, p. 230, for a discussion of this point.

transient analysis. These assumptions are fundamental and should always be kept in mind in relating the theoretical analysis to the practical network. The justification for the assumptions to follow is that they enable an analytical solution while, at the same time, they generally do not impose unrealistic limitations upon the actual network.

**4. Four-terminal Networks.**—A general four-terminal network is shown in Fig. 1. It consists of two input terminals, 1-2, across which a voltage is impressed, and two output terminals, 3-4, across which a voltage appears as a result of the applied voltage.

In general, the network can contain any electrical elements whatsoever in any conceivable configuration. These elements can be either linear or nonlinear and can be lumped or distributed.

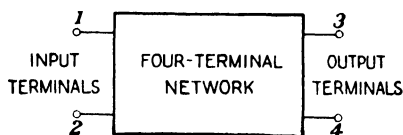


Fig. 1.—A general four-terminal electrical network.

**5. Electrical Elements.**—There are three different types of linear electrical elements. They are commonly known as resistors, inductors, and capacitors. These elements exhibit electrical properties of resistance, inductance, and capacitance, respectively. Strictly speaking, any single practical element exhibits all three electrical properties. For instance, a coil of wire (inductor) consists of inductance of the windings, resistance of the wire, and capacitance among the turns. However, in many instances two of the three constituents of a practical electrical component are negligible compared with the third. In such cases it is often justifiable to neglect all electrical properties except the predominant one. A linear electrical component that exhibits essentially one electrical property is called an *ideal* element.

An ideal resistor is an electrical element in which the ratio potential/current is constant; *i.e.*, the voltage across an ideal resistor is directly proportional to the current flowing through the resistor.

An ideal inductor is an electrical element in which the ratio potential/(rate of change of current) is constant; *i.e.*, the volt-

age across an ideal inductor is directly proportional to the rate of change of current through the inductor.

An ideal capacitor is an electrical element in which the ratio potential/charge is constant; *i.e.*, the voltage across an ideal capacitor is directly proportional to the charge that exists upon the capacitor.

**6. Linear and Nonlinear Elements.**—The behavior of a linear element can be described by means of a linear equation. For example, the behavior of an element that consists solely of resistance is described by Ohm's law,  $e_R = Ri_R$ . This law states that the relationship between the instantaneous voltage  $e_R$  across an ideal resistor equals a constant  $R$  times the instantaneous current  $i_R$  flowing through the resistor.  $R$  is called a *parameter* of the element, and the fact that it is constant means that the element is linear. In general, if the variables involved in describing the electrical properties of an element are graphed and a straight-line relationship exists between the variables, the element is linear. In the case of an ideal inductor the linear equation is  $e_L = L (di_L/dt)$  where  $e_L$ , the instantaneous voltage across the inductor, is a linear function of the instantaneous rate of change of current  $di_L/dt$  through the inductor. In the case of an ideal capacitor the linear equation is  $e_c = q_c/C$  where  $e_c$ , the instantaneous voltage across the capacitor, is a linear function of the instantaneous charge  $q_c$  on the capacitor. The constant parameters of these elements are  $L$  and  $C$ , respectively.

Strictly speaking, no practical electrical element is truly linear, but in many cases the departure from linearity is so slight that it is experimentally impossible to distinguish between the actual element and the assumed linear element. Nonlinearity in an element generally becomes evident when the operating range is large. For example, the voltage across a resistor may be directly proportional to the current through the resistor for low values of current, but if the current is increased to a large enough value, this linear relationship breaks down and the resistance undergoes a change. In some cases an element can be nonlinear before it is subjected to extreme operating ranges. An excellent illustration of this is an iron-core inductor where the presence of a magnetic material introduces a nonlinear relationship between  $e_L$  and  $di_L/dt$ . When elements are non-

linear, obviously their behavior can no longer be described by constant parameters.

There are nonelectrical examples of linear and nonlinear elements. A steel coil spring is ordinarily a linear element because the relationship between the stretching of the spring and the force required for stretching is a straight-line or linear relationship. If too much force is applied, however, the spring becomes permanently deformed and the relationship between stretching force and displacement is no longer constant. To return to the previously mentioned diving-board example, for light-weight men the diving board is a linear element because its elastic limit is not exceeded. However, a heavy diver might crack the board, and the linear relationship between applied force and displacement of the board would no longer exist.

In the networks to be treated in this book all elements (1) are assumed to be linear and (2) are assumed to exist alone as either pure  $R$ , pure  $L$ , or pure  $C$ . The first assumption must always be kept in mind and must be recognized as a limitation. Fortunately, many network elements are approximately linear or their departure from linearity is slight. The second assumption is also applicable to many physical elements. In cases where this assumption is not valid, it is often possible to overcome the difficulty. For instance, a coil may have negligible capacitance between windings, but the coil wire may have appreciable resistance. In this case it is often possible to assume that the coil can be represented by a combination of two parameters, one a pure resistance and the other a pure inductance.

**7. Lumped and Distributed Elements.**—The distinction between lumped and distributed elements is that there are no space considerations in the former, whereas, in the latter, space must be considered. As an example of a lumped element consider a 50-ft. length of resistance wire through which current is flowing. If at any instant the current in one part of the wire is exactly the same as the current in every other part of the wire, then the wire can be replaced by a very small fixed resistor that has a resistance equal to the resistance of the wire, without altering the electrical considerations.

Under certain conditions the current in one part of an element is different from the current in another part of the same physical element. This phenomenon becomes increasingly evident at high

frequencies. In such a case it is not possible to consider the element to be lumped or concentrated at one point, and its distributive or space properties must be taken into account.

All networks in this book consist of lumped parameters. This does not exclude distributed network elements entirely, because in many instances a distributed element can be regarded as consisting of a combination of lumped parameters. Nevertheless, it should be remembered that each time  $R$ ,  $L$ , or  $C$  is used, it represents a linear lumped network parameter.

**8. Disturbing Force.**—All networks to be analyzed in this book are subjected to one particular type of disturbing force which is shown in Fig. 2. It is a single rectangular-pulse voltage.

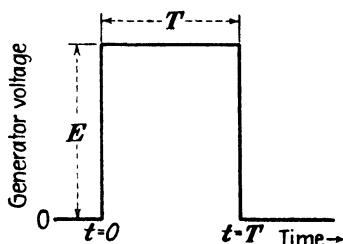


FIG. 2.—Rectangular pulse generator voltage of amplitude  $E$  and duration  $T$ . Zero time is arbitrarily chosen at the instant the pulse voltage changes from zero to  $E$ .

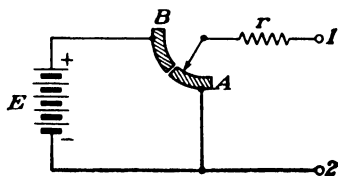


FIG. 3.—Simple switching arrangement which can be used to produce a voltage that is approximately rectangular. The internal generator resistance is represented by  $r$ .

The magnitude of this pulse is any voltage  $E$ , and its duration is  $T$  seconds. It is assumed that up to some arbitrary time that will be called *zero* time, the voltage is zero; that the voltage  $E$  is attained instantaneously at zero time; that during the subsequent time interval  $T$  the voltage is a constant value  $E$ ; and that the voltage becomes zero instantaneously at the end of the time interval  $T$ . Such a pulse cannot be produced in practice. Nevertheless, it is justifiable to consider this ideal pulse because

1. This ideal pulse can be closely attained in practice (refer to Fig. 4).
2. The results are applicable to pulses that are reasonably similar to this ideal pulse.
3. It is convenient to analyze networks that are subjected to such a pulse.

4. The results of analyses based upon such a pulse can indicate a great deal about the general transient behavior of the network.

There are numerous means by which approximately rectangular pulses can be produced. One crude method is to use a battery of voltage  $E$  and a switch. In Fig. 3 the switch is in position  $A$  up to the time  $t = 0$ . Then it is very quickly switched to position  $B$  and held there for a time  $T$ . Then the switch is very quickly returned to position  $A$ . The voltage at the terminals 1-2, which might be the two input terminals of a network, will resemble the ideal pulse assumed in Fig. 2. The internal resistance of the pulse generator is represented by  $r$ . Notice that the resistance across the terminals 1-2 is independent of the position of the switch inasmuch as the battery resistance is assumed to be zero.

A more refined method by which approximately rectangular pulses can be produced is by electronic means. It is possible with ordinary vacuum tubes to produce rectangular pulses which rise to a voltage  $E$  as quickly as one-ten-millionth of a second (0.0000001 sec.) and which become zero, after a duration  $T$ , in a comparable time. The oscillogram in Fig. 4 shows a rectangular pulse that was produced by a vacuum-tube multivibrator circuit.

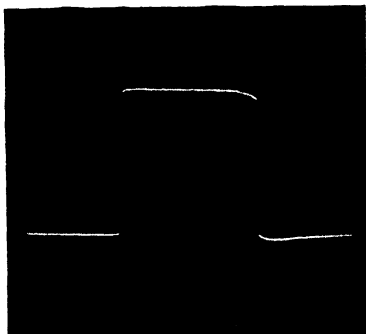


FIG. 4.—Rectangular pulse produced by a vacuum-tube multivibrator circuit.

**9. Statement of Problem.**—It is now possible to state the specific problem to be treated in this book. The general problem in transient analysis is depicted in Fig. 5a, while the simplified problem is given in Fig. 5b. This figure shows that one particular disturbing voltage is to be used, that the internal impedance of the voltage generator consists of pure resistance, and that the network consists of lumped linear parameters only. The question to be answered for various networks is: What voltage appears across the output terminals 3-4 as a result of the application of a single rectangular pulse of voltage  $E$  and duration  $T$  across the input terminals 1-2; in other words, what is the pulse-response characteristic of the network? Since the differential-

equation method of analysis is to be employed, an auxiliary question is: How many different types of networks can be conveniently handled by this method, and what are its limitations?

In some instances a series of pulses, rather than a single pulse, is applied to networks. This analysis is applicable with the provision that the network transient diminishes to a negligible value between generator pulses. In other words, the decay time of the transient must be small compared with the period of the generator pulses.

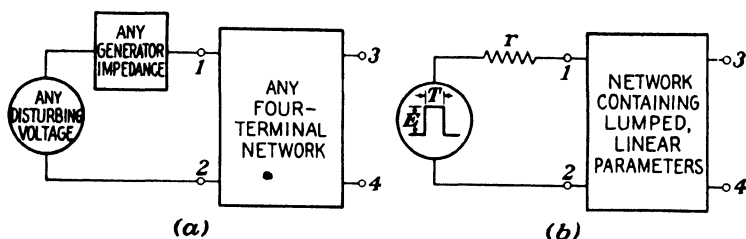


FIG. 5.—Comparison of general pulsed network and simplified network.

In cases where the generator does not contain an internal resistance only, the necessary additional parameters can be connected between the generator and the network.

### ELECTRICAL CONCEPTS

Before proceeding with the analysis, it is well to review some of the basic concepts that will be required. There are very few special ideas necessary. This is one of the outstanding features of the classical method.

**10. Basic Laws.**—Some of the electrical concepts required have already been stated, namely, the laws for voltage across the three network parameters  $R$ ,  $L$ , and  $C$ . These laws can be expressed in several ways. To establish a multiplicity of ways, the definition of current in terms of charge is required. The instantaneous rate of change of charge is equal to the instantaneous current, or mathematically

$$\frac{dq}{dt} = i \quad (1)$$

This can be written

$$q = \int i \, dt \quad (2)$$

Therefore, the three voltage laws can be written

$$e_R = Ri_R = R \frac{dq_R}{dt} \quad (3)$$

$$e_L = L \frac{di_L}{dt} = L \frac{d^2q_L}{dt^2} \quad (4),$$

$$e_C = \frac{q_C}{C} = \frac{1}{C} \int i_C dt \quad (5)$$

where  $e_R$ ,  $e_L$ , and  $e_C$  are instantaneous voltages across the parameters  $R$ ,  $L$ , and  $C$ ,  $i_R$  is the instantaneous current through  $R$ ,  $di_L/dt$  is the instantaneous rate of change of current through  $L$ , and  $q_C$  is the instantaneous charge on  $C$ . These laws are essentially definitions of the parameters  $R$ ,  $L$ , and  $C$ .

**11. Kirchhoff's Laws.**—Kirchhoff's voltage law states that the algebraic sum of all instantaneous voltages around a closed network is zero. Mathematically this law can be stated by the equation

$$e_1 + e_2 + e_3 + \cdots + e_n + \cdots = 0 \quad (6)$$

where  $e_1$ ,  $e_2$ ,  $e_3$ , . . .  $e_n$  are instantaneous voltages that can be either positive (voltage rise) or negative (voltage drop). Kirchhoff's current law states that the algebraic sum of all instantaneous currents at any junction in a network is zero.

**12. Elements in Series.**—It is possible for a single parameter to represent any number of electrical elements in a network. Specifically, when any number of like ideal elements are connected in series, it is possible to represent all of these elements by a single parameter. To demonstrate this, Kirchhoff's laws can be applied to a few illustrative networks.

*Resistors in Series.*—Figure 6a shows a series network in which the instantaneous current is  $i_R$ . The same current flows through each ideal resistor since they are all connected in series. The algebraic sum of the instantaneous voltages around this closed network is zero.

$$\begin{aligned} e_v - i_R R_1 - i_R R_2 - i_R R_3 - i_R R_4 &= 0 \\ \text{or} \quad e_v &= i_R (R_1 + R_2 + R_3 + R_4) \end{aligned}$$

Now, a single resistance

$$R_s = R_1 + R_2 + R_3 + R_4 \quad (7)$$

can replace the series combination without affecting the current flow. Therefore, the network in Fig. 6b is exactly the same as that in Fig. 6a insofar as the generator and network current is concerned. Thus a single resistance  $R_s$  is equivalent to the series resistors  $R_1$ ,  $R_2$ ,  $R_3$ , and  $R_4$  irrespective of the generator voltage.

*Inductors in Series.*—In Fig. 7a a series of ideal inductors is connected across a generator of any instantaneous voltage  $e_u$ . The current in each inductor and the rate of change of current

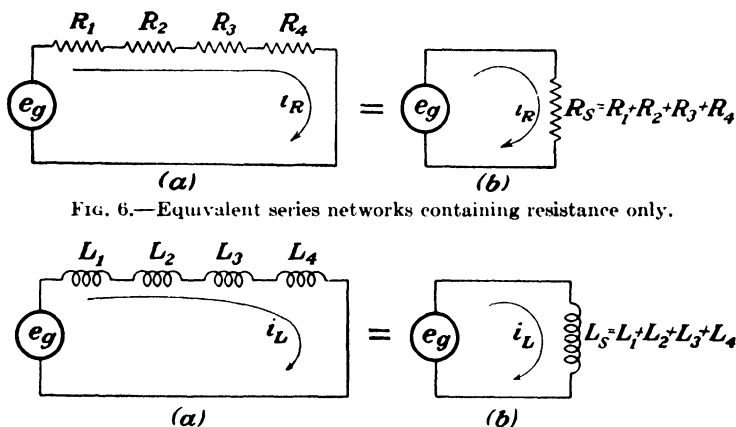


FIG. 6.—Equivalent series networks containing resistance only.

FIG. 7.—Equivalent series networks containing inductance only. Mutual effects are neglected.

in each inductor are the same. Sum the voltages in accordance with Kirchhoff's law.

$$e_u = \frac{di_L}{dt} (L_1 + L_2 + L_3 + L_4)$$

A single inductance

$$L_s = L_1 + L_2 + L_3 + L_4 \quad (8)$$

placed across the generator terminals as shown in Fig. 7b will cause the same instantaneous current to flow and will result in the same rate of change of current as in the case of Fig. 7a. Therefore, the single inductance  $L_s$  is equivalent to the series of inductors, and the networks in Figs. 7a and 7b are equivalent.

*Capacitors in Series.*—When any number of ideal capacitors are connected in series, as indicated in Fig. 8a, there is a single

capacitance that, when connected across the generator, will draw the same instantaneous charge from the generator as the series of capacitors. To demonstrate this, apply Kirchhoff's law to Fig. 8a where the instantaneous charge on each capacitor must be the same.

$$e_g = q_c \left( \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \frac{1}{C_4} \right)$$

From this equation it can be seen that a single capacitance  $C_s$  will draw the same charge from the generator as the series

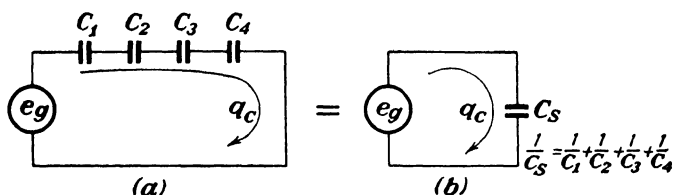


FIG. 8.—Equivalent series networks containing capacitance only. Mutual effects are neglected.

capacitors, no matter what the value of generator voltage, provided

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \frac{1}{C_4} \quad (9)$$

Thus the network in Fig. 8b is equivalent to that in Fig. 8a.

**13. Elements in Parallel.**—In a similar manner it can be proved that a single parameter is equivalent to any number of like elements connected in parallel. For parallel resistors

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots + \frac{1}{R_n} + \cdots \quad (10)$$

For parallel inductors

$$\frac{1}{L_p} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_n} + \cdots \quad (11)$$

For parallel capacitors

$$C_p = C_1 + C_2 + C_3 + \cdots + C_n + \cdots \quad (12)$$

Equations (10), (11), and (12) apply to ideal resistors, inductors, and capacitors. Furthermore, mutual effects sometimes significant in practice are neglected in obtaining these equations.

In general, any combination (either series, parallel, or series-parallel) of *like* elements can be replaced by a single equivalent

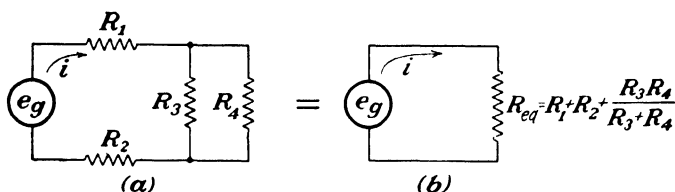


FIG. 9.—Equivalent series network for a series-parallel network containing resistance only.

parameter. For instance, the network in Fig. 9a can be replaced by a single equivalent resistance. The value of this resistance is

$$R_{eq} = R_1 + R_2 + \frac{R_3 R_4}{R_3 + R_4}$$

**14. Thévenin's Theorem.**—A very useful network theorem can be explained with the aid of Fig. 10. This theorem is a special form of Thévenin's theorem and is not stated here in its most general terms.<sup>1</sup>

In Fig. 10a a generator of instantaneous voltage  $e_g$  and internal resistance  $r$  is delivering current to a simple network that

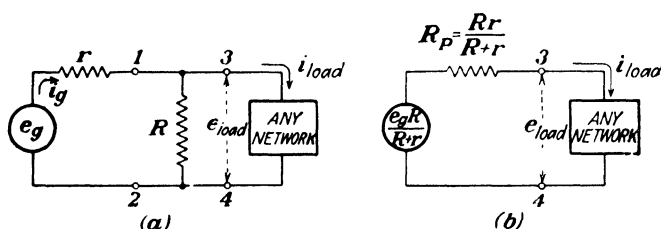


FIG. 10.—Equivalent networks as obtained by Thévenin's theorem.

consists of a resistance  $R$ . Across the output of this network, a general network with any properties whatsoever is connected. This theorem enables  $e_g$ ,  $r$ , and  $R$  to be replaced by an equivalent generator and an equivalent series resistance, thus simplifying the network. The equivalent generator in series with the equivalent resistance will deliver the same current to the

<sup>1</sup> For a general statement of Thévenin's theorem refer to W. L. Everitt, "Communication Engineering," 2d ed., p. 47, McGraw-Hill Book Company, Inc., New York, 1937.

load as the actual network, will produce the same voltage across terminals 3-4 as the actual network both when the load is connected and disconnected, and will present the same resistance across terminals 3-4 as the actual network. In other words, the equivalent series network is exactly the same as the actual network insofar as the output terminals 3-4 are concerned. To find the value of the equivalent generator voltage and equivalent series resistance disconnect the load from terminals 3-4. The voltage across 3-4 with the load disconnected is the voltage of the equivalent generator, and the resistance across 3-4 with the generator short-circuited is the equivalent series resistance. Figure 10*b* shows the equivalent series network. The equivalent generator voltage is  $e_g R / (R + r)$  and is obtained by disconnecting the load and noticing that the current flowing through  $R$  is  $e_g / (R + r)$ . This current multiplied by  $R$  is the voltage across 3-4. The equivalent resistance is obtained by removing the load and replacing the generator by a short circuit. The resistance across 3-4 is then  $R$  in parallel with  $r$ , or a single resistance of value  $R_p = Rr / (R + r)$ .

As a simple demonstration of the equivalence of the two networks, suppose the load is a resistance  $R_1$ .

*Actual Network.*—In the actual network the current that flows from the generator is

$$i_g = \frac{e_g}{r + \frac{RR_1}{R + R_1}}$$

This current produces a voltage drop across  $r$  equal to

$$i_g r = \frac{e_g r}{r + \frac{RR_1}{R + R_1}}$$

Therefore, the voltage across 1-2 and across 3-4 is

$$e_{\text{load}} = e_g - i_g r = \frac{e_g RR_1}{Rr + R_1 r + RR_1} \quad (13)$$

The current flowing through the load in the actual network is consequently

$$i_{\text{load}} = \frac{e_{\text{load}}}{R_1} = \frac{e_g R}{Rr + R_1 r + RR_1} \quad (14)$$

The current that would flow in the actual network if 3-4 were short-circuited is

$$i_{sc} = \frac{e_g}{r} \quad (15)$$

As stated before, the voltage across 3-4 with the load removed is

$$e_{oc} = \frac{e_g R}{R + r} \quad (16)$$

and the resistance across 3-4 with  $R_1$  removed and the generator short-circuited is

$$R_P = \frac{Rr}{R + r} \quad (17)$$

*Equivalent Network.*—Now, in the equivalent network the load current is

$$i_{load} = \frac{\frac{e_g R}{R + r}}{R_1 + \frac{Rr}{R + r}} = \frac{e_g R}{Rr + R_1 r + RR_1}$$

which is exactly the same as Eq. (14) for load current in the actual network. The load voltage is

$$e_{load} = i_{load} R_1 = \frac{e_g RR_1}{Rr + R_1 r + RR_1}$$

which is exactly the same as Eq. (13). In the equivalent network, the voltage across 3-4 with  $R_1$  disconnected is equal to the equivalent generator voltage

$$e_{oc} = \frac{e_g R}{R + r}$$

which is exactly the same as Eq. (16). In the equivalent network the resistance across 3-4 with  $R_1$  removed and the generator short-circuited is

$$R_P = \frac{Rr}{R + r}$$

which is the same as Eq. (17).

These two networks are exactly equivalent as far as the output terminals 3-4 are concerned, and the equivalence is

valid irrespective of the value of the generator voltage and of the value of  $R_1$ . However, it is important to notice that they are not equivalent as far as the generators are concerned. When  $R_1$  is removed, the generator current in the actual network is  $e_g/(R + r)$ , while in the equivalent network the generator current is zero. The generator currents are also different when the load is connected.

**15. Dimensions of Electrical Quantities.**—Equations (3); (4), and (5), which define  $R$ ,  $L$ , and  $C$ , also define the units of  $R$ ,  $L$ , and  $C$  in terms of the units of voltage and charge. Equation (1) defines the unit of current in terms of charge. If the units of voltage, charge, and time are taken as basic units, then the units of current, resistance, inductance, and capacitance can be stated in terms of voltage, charge, and time. From Eq. (1) the unit of current is charge/time or symbolically  $q/t$ , where  $t$  denotes time. From Eq. (3) the unit of resistance is

$$\frac{\text{voltage}}{\text{current}} = \frac{\text{voltage} \times \text{time}}{\text{charge}} \quad \text{or} \quad \frac{Vt}{q}.$$

From Eq. (4) the unit of inductance is

$$\frac{\text{voltage} \times \text{time}}{\text{current}} = \frac{\text{voltage} \times (\text{time})^2}{\text{charge}} \quad \text{or} \quad \frac{Vt^2}{q}.$$

From Eq. (5) the unit of capacitance is  $\frac{\text{charge}}{\text{voltage}}$  or  $\frac{q}{V}$ . These dimensional relationships enable others to be derived. For example:

$$\begin{aligned} RC: \frac{Vt}{q} \times \frac{q}{V} &= t \rightarrow \text{time} \\ \frac{L}{R}: \frac{Vt^2/q}{Vt/q} &= t \rightarrow \text{time} \\ \sqrt{LC}: \sqrt{\frac{L}{R}} \times RC &= \sqrt{t^2} \rightarrow \text{time} \\ \sqrt{\frac{L}{C}}: \sqrt{\frac{Vt^2/q}{q/V}} &= \sqrt{\frac{V^2t^2}{q^2}} = \frac{Vt}{q} \rightarrow \text{resistance} \\ R^2C: \frac{V^2t^2}{q^2} \times \frac{q}{V} &= \frac{Vt^2}{q} \rightarrow \text{inductance} \\ \frac{L}{R^2}: \frac{Vt^2/q}{V^2t^2/q^2} &= \frac{q}{V} \rightarrow \text{capacitance} \end{aligned}$$

TABLE I.—PRIMARY SYMBOLS

$a$	constant
$A$	constant
$\alpha$	constant = $\tanh^{-1} \frac{\sqrt{M^2 - N}}{M}$
$b$	constant
$B$	constant
$\beta$	constant = $\tan^{-1} \frac{\sqrt{N - M^2}}{M}$
$c$	constant
$C$	capacitance parameter
$\epsilon$	constant = 2.71828 . . .
$e$	instantaneous output voltage
$e_E$	instantaneous output voltage during the time that the generator voltage is $E$
$e_0$	instantaneous output voltage after the generator voltage has become zero
$e_g$	instantaneous generator voltage of any shape
$E$	generator-pulse voltage
$F$	filtering ratio
$G$	constant
$H$	constant
$i$	instantaneous current
$j$	constant = $\sqrt{-1}$
$K$	arbitrary constant in the solution of a differential equation
$L$	inductance parameter
$M_{12} = M_{21}$	mutual inductance parameter
$M$	constant = $\frac{R + r}{2L}$
$\mu$	amplification factor
$N$	constant = $1/LC$
$p$	instantaneous power
$P$	constant
$q$	instantaneous charge
$r$	internal generator resistance
$R$	resistance parameter
$S$	constant
$t$	time
$T$	generator-pulse width
$V$	voltage
$w$	instantaneous total energy
$\omega$	$2\pi$ times the frequency

These combinations of  $R$ ,  $L$ , and  $C$  arise in the analysis, and it will prove useful to remember the dimensional relationships.

### NOTATION

The notation used throughout this text is consistent, and an attempt has been made to use standard symbols wherever possible, but above all to use notation that is not meaningless. A list of primary symbols used and the meanings of the symbols are shown in Table I.

Subscripts are used frequently throughout, and in general their meaning should be self-explanatory. However, in the case of the subscript  $T$  it must be understood that it refers to the last instant at which the rectangular generator pulse exists.

### Problems

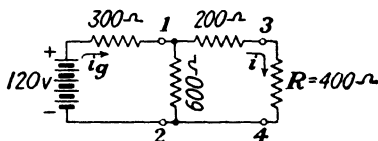
**Prob. 1.** Three resistors have values  $R_1 = 600$  ohms,  $R_2 = 200$  ohms, and  $R_3 = 150$  ohms.

- When  $R_1$ ,  $R_2$ , and  $R_3$  are connected in parallel, what is the equivalent resistance of the combination?
- When  $R_1$ ,  $R_2$ , and  $R_3$  are connected in series, what is the equivalent resistance of the combination?
- When  $R_1$  and  $R_2$  are connected in parallel and this combination is connected in series with  $R_3$ , what is the equivalent resistance?

**Prob. 2.** Two inductors  $L_1$  and  $L_2$  are connected in parallel. What is the ratio  $L_2/L_1$  if the equivalent inductance of the parallel combination is equal to 95 per cent of  $L_1$ ?

**Prob. 3.** Three capacitors  $C_1$ ,  $C_2$ , and  $C_3$  are connected in series and the equivalent capacitance is  $0.08 \mu\text{f}$ . When they are connected in parallel, the equivalent capacitance is  $1.16 \mu\text{f}$ . When  $C_1$  and  $C_3$  are connected in series and this combination is connected in parallel with  $C_2$ , the equivalent capacitance is  $0.32 \mu\text{f}$ . What are the values of  $C_1$ ,  $C_2$ , and  $C_3$ ?

**Prob. 4.** Refer to the network below.



- Find the currents  $i_g$  and  $i$  by means of Kirchhoff's laws.
- Remove  $R$  from the terminals 3-4 and determine the equivalent series network by means of Thévenin's theorem.
- When  $R$  is connected to the equivalent series network, what is the current through  $R$ ? What is the current through the equivalent generator when  $R$  is connected?

**Prob. 5.** An electric-light bulb is turned on at a time corresponding to zero in the table below, and the bulb temperature is measured at different times. The light is then turned off, and again temperatures are measured at various times. This is done for three different time-on periods.

Test 1		Test 2		Test 3	
Temp., °F.	Elapsed time, min.	Temp., °F.	Elapsed time, min.	Temp., °F.	Elapsed time, min.
70	0 on	70	0 on	70	0 on
89	1	90	1	88	1
100	2 off	104	2	115	3
96	3	126	4	140	6
91	5	144	7 off	150	8
83	9	129	10	165	15
77	14	116	13	169	25
73	19	107	16	170	30
71	24	96	20	170	35 off
70	30	87	24	152	36
70	35	80	29	137	37
		72	40	114	39
				87	44
				71	60

Make a graph of these data using time as the abscissa and temperature as the ordinate, and then answer the following questions:

- What is the steady-state value of temperature in each test during the time that the bulb is on?
- What is the steady-state value of temperature in each test during the time after the bulb is turned off?
- In test 3 what percentage of the *total increase* in temperature has taken place in the first 5 min.?
- In test 3 what percentage of the *total decrease* in temperature has taken place 5 min. after the bulb is turned off?
- What is the maximum time that the bulb can remain on before its temperature exceeds 158°F.? (Assume its temperature is 70°F. at the instant it is turned on.)
- On the temperature-versus-time graph, draw curves for each of the tests showing light intensity versus time. (Use an arbitrary scale for the light intensity, which reaches maximum intensity in a fraction of a second.)

**Prob. 6.** What are the dimensions, in terms of either resistance, inductance, capacitance, or time *only*, of the following:

(a)  $\frac{R^3 C^2}{L}$

(d)  $\frac{RC}{L}$

(b)  $\sqrt{\frac{L^2}{R^4 C^3}}$

(e)  $\frac{L^2}{R^4 C}$

(c)  $\frac{L^2}{R^2 C}$

(f)  $\sqrt{LR^2 C}$

**Prob. 7.** A variable voltage is applied to a resistor and measurements of applied voltage and the corresponding current through the resistor are taken. The readings are given in the table below.

Voltage, Volts	Current, Milliamperes
0	0
3.5	2.8
14.5	11.6
27.5	22.0
34.2	27.3
61.5	49.0
78.0	61.1
84.3	65.6
95.9	73.0
112	81.4
128	88.0
149	95.9
161	99.0

(NOTE: A graph of *resistance* versus current will be helpful in this problem.)

- Over what range of current is the resistor "linear" within 1 per cent?
- What is the value of the "linear" resistance?
- If the voltage is increased without limit, what will be the ultimate value of the resistance?

## CHAPTER II

### DIFFERENTIAL EQUATIONS AND HYPERBOLIC FUNCTIONS

This chapter is intended to equip the reader with the mathematics required for the analysis of the linear networks to be treated in this book. An important function of this chapter is to isolate from a voluminous mathematical subject the small amount of material which is required for the classical method.<sup>1</sup> In doing this, the mathematical ideas are presented with physical interpretations in terms of the particular study of pulsed linear networks.

#### DIFFERENTIAL EQUATIONS

The primary mathematical tool of analysis used in the classical method is differential equations. A differential equation is simply an equation that involves differentials or derivatives.

**1. General Differential Equation.**—When Kirchhoff's voltage law is applied to a network, an equation involving differentials generally results. This is because the voltage across a network element is defined in terms of differentials. For instance, the voltage across a resistance is  $R(dq/dt)$ , and the voltage across an inductance is  $L(d^2q/dt^2)$ .

The most general differential equation that can result from the application of Kirchhoff's laws to a linear network of any configuration is of the form shown in Eq. (18).

$$b_n \frac{d^n q}{dt^n} + \cdots + b_2 \frac{d^2 q}{dt^2} + b_1 \frac{dq}{dt} + b_0 q = e_g \quad (18)$$

where  $n$  is any positive integer;  $b_n, \dots, b_2, b_1, b_0$  are constants determined by the network parameters;  $q$  is the instantaneous charge that is a function of time  $t$  only; and  $e_g$  is the instantaneous generator voltage that is a function of time only. Equation (18)

<sup>1</sup> For a more detailed treatment of differential equations refer to A. Murray, "Differential Equations," Longmans, Green and Company, or any other standard differential equations text.

is called a homogeneous, linear, first-degree equation of the  $n^{\text{th}}$  order with constant coefficients. Notice that there are no terms of higher degree than the first; *i.e.*, no differentials are squared, cubed, etc., and there are no products of variables such as  $(dq/dt)(d^2q/dt^2)$  or  $q(dq/dt)$ . All of these restrictions make it one of the simplest types of differential equations.

**2. Restrictions on General Differential Equation.**—Two additional restrictions can be placed upon Eq. (18) to make it representative of a typical equation resulting from the application of Kirchhoff's voltage law to the linear networks in this book. One of these restrictions is that the highest order shall be the second order. Upon superficial examination, this may appear to be a rather drastic limitation but fortunately it does not exclude many practical networks. As a matter of fact, no series networks whatsoever are excluded; however, many series-parallel networks give rise to differential equations of higher order than the second. Such networks are not treated in this book because their solution is usually too involved when the classical method is used.

The second additional restriction involves  $e_v$ . In all networks  $e_v$  is to be a single rectangular pulse, Fig. 2. One means of expressing such a voltage as a continuous function of time is by use of the Fourier integral.<sup>1</sup> However, when the single rectangular pulse is expressed as a discontinuous function of time, it can be handled very simply. The equation for  $e_v$  during the interval  $t = 0$  to  $t = T$  is  $e_v = E$ , where  $E$  is a constant. And  $e_v$  for all values of  $t$  outside the interval  $t = 0$  to  $t = T$  is a constant equal to zero. Therefore, the restriction placed upon  $e_v$  is that it be a constant with the understanding that the equation hold for restricted values of time  $t$ . In most cases this means that two differential equations will be

<sup>1</sup> The Fourier integral that expresses a single-pulse voltage  $E$  and duration  $T$  as a continuous function of time is

$$e_v = \frac{ET}{2\pi} \int_{-\infty}^{+\infty} \frac{\sin \omega T/2}{\omega T/2} e^{j\omega t} d\omega$$

where  $\omega = 2\pi \times \text{frequency}$  and  $j = \sqrt{-1}$ . From the Fourier integral viewpoint a single rectangular pulse is regarded as being the sum of an infinite number of sinusoidal waves. These sine waves completely cancel one another from  $t = -\infty$  to  $t = +\infty$  except during the pulse interval. During the pulse interval, the sum of the sine waves is a constant value  $E$ .

required, one that applies during the generator pulse and one that applies at all other times.

Incorporating these two restrictions, Eq. (18) becomes

$$b_2 \frac{d^2q}{dt^2} + b_1 \frac{dq}{dt} + b_0 q = E \quad (19)$$

where the constant  $E$  has a value of zero if  $t$  lies outside the interval  $t = 0$  to  $t = T$ . Equation (19), then, is the form of the most complicated differential equation that will be solved in this book.

Each term in Eq. (19) represents a voltage. The first term is of the form of a voltage across an inductance,  $L(di/dt)$ ; the second term is of the form of a voltage across a resistance,  $Ri$ ; the third term is of the form of a voltage across a capacitance,  $q/C$ ; and  $E$  is the generator or applied voltage. It follows that  $b_2$  is an inductance parameter,  $b_1$  is a resistance parameter, and  $b_0$  is the reciprocal of a capacitance parameter.

**3. Particular Integral.**—The problem presented by the differential equation is to find values of  $q$  that, when substituted into the differential equation, result in an identity. Any values of  $q$  that result in an identity are called solutions of the differential equation. Inspection of Eq. (19) reveals that if  $q = E/b_0 = \text{constant}$ , then Eq. (19) becomes an identity  $E \equiv E$ , because  $d^2q/dt^2 = dq/dt = 0$  when  $q$  is constant. This constant value of  $q$  that satisfies the differential equation is called, mathematically, the *particular integral*. In terms of the network to which the differential equation applies, the particular integral has important significance and is known as the *steady-state term*. The particular integral gives the steady-state condition because it is a solution when  $d^2q/dt^2$  and  $dq/dt$  are zero, i.e., when  $q$  is not varying.

**4. Complementary Function.**—The particular integral is not the only possible value of  $q$  that will satisfy the differential equation. There is another value of  $q$ , called the *complementary function*, that when added to the particular integral and substituted into the differential equation results in an identity. When referring to a physical network, the complementary function is called the *transient term*. This nomenclature is appropriate because the complementary function is a function of time, as is the transient in a physical network.

For a linear, second-order, first-degree equation with constant coefficients it can be demonstrated that the complementary function is always of the form

$$q' = K\epsilon^{-At} \quad (20)$$

where  $K$  is an arbitrary constant, and  $A$  is a constant determined by the network parameters. The manner in which  $K$  and  $A$  can be evaluated is indicated below.

The complete solution of the differential equation consists of the sum of the particular integral and the complementary function.

$$q = \frac{E}{b_0} + K\epsilon^{-At} \quad (21)$$

To demonstrate that this is a solution, differentiate

$$\frac{dq}{dt} = -AK\epsilon^{-At}; \quad \frac{d^2q}{dt^2} = A^2K\epsilon^{-At}$$

and substitute into the original differential equation.

$$b_2(A^2K\epsilon^{-At}) + b_1(-AK\epsilon^{-At}) + b_0\left(\frac{E}{b_0} + K\epsilon^{-At}\right) = E$$

Collect all terms involving  $K\epsilon^{-At}$ .

$$K\epsilon^{-At}(b_2A^2 - b_1A + b_0) + E = E$$

Now this equation is an identity, and therefore Eq. (21) is a solution, if

$$K = 0$$

or if

$$\epsilon^{-At} = 0$$

or if

$$(b_2A^2 - b_1A + b_0) = 0$$

The first two possibilities are trivial since in either one the complementary function, Eq. (20), is always zero. The third possibility is significant and determines the value of  $A$ , in terms of the network parameters, that is required for the exponent of the complementary function. This equation for  $A$  is called the *auxiliary* equation and can be solved by the quadratic formula. There are two values of  $A$  that satisfy the auxiliary equation. They are

$$A_1 = \frac{b_1 + \sqrt{b_1^2 - 4b_0b_2}}{2b_2} \quad (22)$$

$$A_2 = \frac{b_1 - \sqrt{b_1^2 - 4b_0b_2}}{2b_2} \quad (23)$$

It should be clear that if either of the above values of  $A$  is used as the exponent of the complementary function, an identity will result upon substitution of the sum of the complementary function and the particular integral for  $q$  in the original differential equation.

Because  $A$  can have two different values that reduce the differential equation to an identity, the whole complementary function consists of two terms, each of which is the same form as Eq. (20); *i.e.*, the complementary function is

$$q' = K_1 e^{-A_1 t} + K_2 e^{-A_2 t} \quad (24)$$

The fact that the complementary function consists of the sum of these two terms can be shown by substituting once again into the original equation. When this is done, the following result is obtained:

$$K_1 e^{-A_1 t} (b_2 A_1^2 - b_1 A_1 + b_0) + K_2 e^{-A_2 t} (b_2 A_2^2 - b_1 A_2 + b_0) + E = E$$

Each of the terms in the brackets is zero for the values of  $A_1$  and  $A_2$  given by Eqs. (22) and (23), and therefore this expression is an identity. Notice that the values of  $K_1$  and  $K_2$  have no bearing on the fact that

$$q = \frac{E}{b_0} + K_1 e^{-A_1 t} + K_2 e^{-A_2 t}$$

is a complete solution of the differential equation. This is why  $K_1$  and  $K_2$  are called, mathematically, *arbitrary* constants. It is very important to note that if  $E$  is set equal to zero in Eq. (19), the complementary function will be the complete solution of the reduced differential equation.

**5. Arbitrary Constants.**—Although the constants  $K_1$  and  $K_2$  are arbitrary in the sense that the complementary function is determined irrespective of their values, these constants are not arbitrary in a physical network. They are uniquely determined in a physical network by so-called *initial conditions*, or the conditions that exist at the first instant the differential equation applies. Numerous examples of the evaluation

of these constants will be found later on. For instance, refer to Chap. III, page 41.

In general, the number of arbitrary constants contained in the solution of a differential equation is equal to the order of the equation, and any solution that does not contain the required number of arbitrary constants is not a complete solution. In the case of Eq. (19), a second-order equation, two arbitrary constants must appear in the complete solution.

**6. Roots of Auxiliary Equation.**—When the roots of the auxiliary equation given by Eq. (22) and (23) are real, *i.e.*,  $b_1^2 > 4b_0b_2$ , the foregoing solution applies. It is possible in some networks for the roots to be equal or even imaginary. These eventualities are worthy of consideration.

*Equal Roots.*—When the roots are equal,  $b_1^2 = 4b_0b_2$  and  $A_1 = A_2 = b_1/2b_2$ . From Eq. (24) this means that the complementary function is

$$q' = (K_1 + K_2)\epsilon^{-\frac{b_1}{2b_2}t} = K_3\epsilon^{-\frac{b_1}{2b_2}t}$$

and contains only one arbitrary constant. Since the particular integral has no arbitrary constants, this cannot be the complete solution.

To obtain the complete solution, consider the case where the two roots of the auxiliary equation differ by a small amount  $a$ , and then allow this small difference to approach zero. Suppose that

$$A_1 = A_2 + a$$

The complementary function will be

$$q' = K_1\epsilon^{-A_1t} + K_2\epsilon^{-A_2t}$$

which can be written

$$q' = \epsilon^{-A_2t}(K_1\epsilon^{-at} + K_2)$$

The next step is to expand  $\epsilon^{-at}$  in an exponential series.<sup>1</sup> The

<sup>1</sup> Granville, Smith, and Longley, "Elements of the Differential and Integral Calculus," Ginn and Company, or any other standard calculus book.

$$\epsilon^{-at} = 1 - at + \frac{a^2t^2}{2} - \frac{a^3t^3}{6} + \dots$$

complementary function then becomes

$$q' = e^{-A_1 t} \left[ (K_1 + K_2) - K_1 a t + \frac{K_1 a^2 t^2}{2!} - \frac{K_1 a^3 t^3}{3!} + \dots \right]$$

Now let  $a$  become smaller and smaller. All terms involving  $a^2$ ,  $a^3$ ,  $\dots$  will become negligible, and the complementary function becomes

$$q' = e^{-A_1 t} (K_3 - K_4 t)$$

where  $K_3 = K_1 + K_2$  and  $K_4 = K_1 a$ .  $K_1$  and  $a$  are chosen so that  $K_4$  is finite as  $a$  approaches zero. This complementary function contains two arbitrary constants. Therefore, the complete solution for the case where the auxiliary equation has equal roots is

$$q = \frac{E}{b_0} + (K_3 - K_4 t) e^{-\frac{b_1}{2b_0} t} \quad (25)$$

It will be found that an identity results when this value of  $q$  is substituted into Eq. (19), assuming  $A_1 = A_2 = b_1/2b_0$ . This verifies the solution.

**Imaginary Roots.**—If  $b_1^2 < 4b_0b_2$ , then the roots of the auxiliary equation are imaginary. Refer to Eqs. (22) and (23). This case will not be considered in detail here but will be worked out when it appears in the network analysis.<sup>1</sup> Suffice it to say at this time that when the roots are imaginary, the complementary function can be placed conveniently into trigonometric form.

In general, notice that the form of the solution depends upon the roots of the auxiliary equation insofar as they are real and unequal, real and equal, or imaginary; this in turn depends upon the values of the network parameters  $b_2$ ,  $b_1$ , and  $b_0$ .

**7. First-order Differential Equation.**—A special form of Eq. (19) results if, for instance,  $b_2$  is zero. If this is the case, Eq. (19) becomes

$$b_1 \frac{dq}{dt} + b_0 q = E \quad (26)$$

Another example is if  $b_0$  is zero. Equation (19) then becomes

$$b_2 \frac{di}{dt} + b_1 i = E \quad (27)$$

<sup>1</sup> For example, see Chap. V, p. 122.

because 
$$\frac{dq}{dt} = i$$

and 
$$\frac{d^2q}{dt^2} = \frac{di}{dt}$$

Each of these equations is a first-order equation because no higher derivative than the first appears in them.

The solution of an equation of this type is similar to that of the second-order type; namely, it consists of the sum of the particular integral (steady-state term) and the complementary function (transient term). First-order equations can result from electrical networks, so their solution is of interest. For the first-order type of equation, only one arbitrary constant appears in the complete solution.

*Particular Integral.*—By inspection, the particular integral of Eq. (27) is  $i = E/b_1$  and is obtained by setting  $di/dt = 0$ .

*Complementary Function.*—The complementary function is of the form

$$i' = K\epsilon^{-At}$$

and the complete solution is

$$i = \frac{E}{b_1} + K\epsilon^{-At}$$

Differentiate to find the auxiliary equation.

$$\frac{di}{dt} = -AK\epsilon^{-At}$$

Substitute into Eq. (27).

$$b_2(-AK\epsilon^{-At}) + b_1\left(\frac{E}{b_1} + K\epsilon^{-At}\right) = E$$

Group all terms containing  $K\epsilon^{-At}$ .

$$K\epsilon^{-At}(-Ab_2 + b_1) + E = E$$

The auxiliary equation in this case is

$$-Ab_2 + b_1 = 0$$

The condition on  $A$ , then, for this to be an identity is

$$A = \frac{b_1}{b_2}$$

Therefore, the complete solution becomes

$$i = \frac{E}{b_1} + K\epsilon^{-\frac{b_1}{b_2}t} \quad (28)$$

*Separation of Variables.*—The general method of determining the particular integral and the complementary function is not the only means by which a solution for the first-order type of equation can be found. In fact, a shorter method is feasible in this case. Examination of Eq. (27) shows that integration is possible if the variables  $i$  and  $t$  are separated. This means that the solution can be found in essentially one step. Rearrange Eq. (27) so that the variables are separated on either side of the equation.

$$\frac{di}{E - b_1 i} = \frac{dt}{b_2}$$

Integrate both sides of the equation, and the solution is obtained.

$$-\frac{1}{b_1} \ln (E - b_1 i) = \frac{t}{b_2} + K_1$$

$K_1$  is a constant of integration. This equation can be written

$$\ln K_2(E - b_1 i) = -\frac{b_1}{b_2} t$$

Convert this solution to the exponential form in order to solve for  $i$ .<sup>1</sup>

$$K_2(E - b_1 i) = \epsilon^{-\frac{b_1}{b_2}t}$$

Solve for  $i$ .

$$i = \frac{E}{b_1} - \frac{1}{b_1 K_2} \epsilon^{-\frac{b_1}{b_2}t}$$

This is exactly the same solution as Eq. (28), which was obtained by the general method. The relationship between the arbitrary constants in the two cases is  $K = -1/b_1 K_2$ .

**8. Summary.**—To condense the general ideas behind the differential equations to be used in this book, the following tabulation of significant features may be helpful:

<sup>1</sup> The general equivalence of logarithmic and exponential equations can be stated as follows: if  $\log_a b = c$ , then  $a^c = b$ . A conventional shorthand notation for  $\log_e$  is  $\ln$ .

1. Differential equations arise when Kirchhoff's voltage law is applied to a pulsed network.

2. The most complicated differential equation to be encountered is a linear, second-order, first-degree equation with constant coefficients and with a second member that is constant.

3. The solution of this differential equation is characterized by

- a. The particular integral or steady-state term that can be obtained by setting all derivatives equal to zero and solving for the remaining variable.
- b. The complementary function or transient term that is a solution of the differential equation when the second member is set equal to zero.
- c. The complete solution that is the sum of the above.
- d. Two arbitrary constants that can be evaluated from initial conditions.

4. The form of the solution depends upon the roots of the auxiliary equation, which in turn depend upon the values of the network parameters.

5. The first-order differential equation is a special case and can be solved by direct integration. Its solution also consists of a particular integral and complementary function, but it contains only one arbitrary constant.

### HYPERBOLIC FUNCTIONS

It has been indicated that the complementary function of a linear, second-order, first-degree equation with constant coefficients and constant second member can be of the form

$$q' = K_1 e^{-A_1 t} + K_2 e^{-A_2 t}$$

When such a form of the complementary function arises (and  $A_1$  and  $A_2$  are real numbers), it is usually convenient to express it as a hyperbolic function. Because hyperbolic functions are to be used, it is advisable to understand their significance. It is the purpose of this section to introduce the elements of hyperbolic functions, and to consider some of their properties that will be useful in the analysis of pulsed linear networks.

**9. Definitions.**—The defining relations for hyperbolic functions in terms of exponential functions are

$$\sinh At = \frac{e^{At} - e^{-At}}{2}$$

$$\cosh At = \frac{e^{At} + e^{-At}}{2}$$

$$\tanh At = \frac{\sinh At}{\cosh At} = \frac{e^{At} - e^{-At}}{e^{At} + e^{-At}} = \frac{1 - e^{-2At}}{1 + e^{-2At}}$$

To aid in understanding how these hyperbolic functions depend

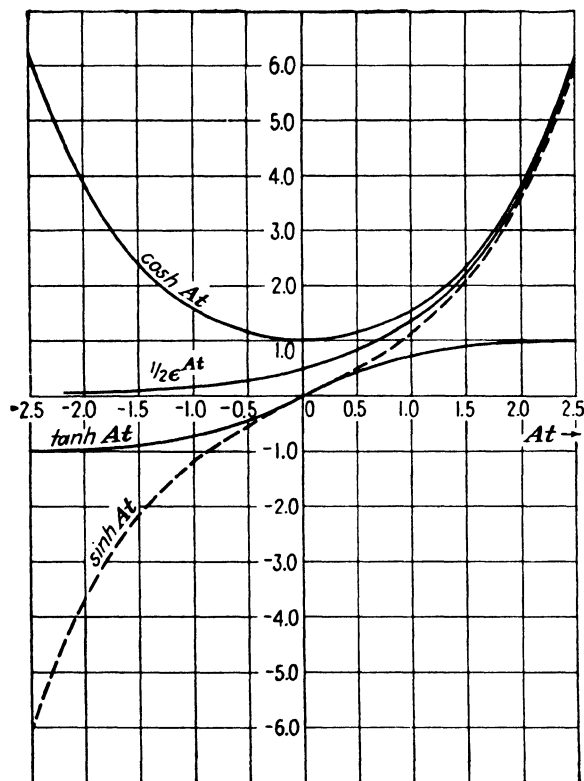


FIG. 11.—Graphical representation of three hyperbolic functions. The curve of  $\frac{1}{2}e^{At}$  is included to show its relationship to  $\sinh At$  and  $\cosh At$  as  $+At$  increases.

upon the value of  $t$ , the graphical representations in Fig. 11 will be helpful. Several pertinent features of these curves are

1.  $\sinh At$  is zero at  $t = 0$ .
2.  $\cosh At$  is 1.0 at  $t = 0$ .
3.  $\tanh At$  is zero at  $t = 0$ .
4. Both  $\sinh At$  and  $\cosh At$  approach  $\frac{1}{2}e^{At}$  as  $At$  becomes a large positive number.
5.  $\tanh At$  approaches  $\pm 1.0$  as  $t$  becomes large.

A table of values of the hyperbolic functions is given in Appendix I.

**10. Hyperbolic Identities.**—Hyperbolic identities can be developed from the three defining relations. Some examples of the method used in proving identities are presented below.

*Identity 1.*  $\sinh (-At) \equiv -\sinh At$

$$\text{Proof: } \sinh (-At) = \frac{e^{-At} - e^{At}}{2} = -\frac{e^{At} - e^{-At}}{2} = -\sinh At$$

*Identity 2.*  $\cosh (-At) \equiv \cosh At$

$$\text{Proof: } \cosh (-At) = \frac{e^{-At} + e^{At}}{2} = \frac{e^{At} + e^{-At}}{2} = \cosh At$$

*Identity 3.*  $\tanh (-At) \equiv -\tanh At$

$$\text{Proof: } \tanh (-At) = \frac{\sinh (-At)}{\cosh (-At)} = \frac{-\sinh At}{\cosh At} = -\tanh At$$

*Identity 4.*  $\cosh^2 At - \sinh^2 At \equiv 1$

$$\begin{aligned} \text{Proof: } \cosh^2 At &= \left( \frac{e^{At} + e^{-At}}{2} \right)^2 = \frac{e^{2At} + 2e^{At}e^{-At} + e^{-2At}}{4} \\ &= \frac{1}{2} + \frac{e^{2At} + e^{-2At}}{4} = \frac{1}{2} + \frac{1}{2} \cosh 2At \end{aligned}$$

$$\begin{aligned} \sinh^2 At &= \left( \frac{e^{At} - e^{-At}}{2} \right)^2 = \frac{e^{2At} - 2e^{At}e^{-At} + e^{-2At}}{4} \\ &= -\frac{1}{2} + \frac{e^{2At} + e^{-2At}}{4} = -\frac{1}{2} + \frac{1}{2} \cosh 2At \end{aligned}$$

$$\begin{aligned} \cosh^2 At - \sinh^2 At &= \frac{1}{2} + \frac{1}{2} \cosh 2At + \frac{1}{2} - \frac{1}{2} \cosh 2At \\ &= \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

Many other identities can be found in a similar manner. It is generally advisable to verify all new hyperbolic identities by substituting the exponential definitions.

**11. A Special Identity.**—An identity that will be used in the network analysis to follow is

$$P \sinh At \pm S \cosh At \equiv \sqrt{P^2 - S^2} \sinh \left( At \pm \tanh^{-1} \frac{S}{P} \right) \quad (29)$$

where  $P$  and  $S$  are constants. The proof of this identity follows: Suppose that

$$P \sinh At \pm S \cosh At \equiv G \sinh (At \pm H) \quad (30)$$

where the constants  $G$  and  $H$  are to be determined. It is necessary to introduce a new hyperbolic identity.

$$G \sinh (At \pm H) \equiv G(\sinh At \cosh H \pm \cosh At \sinh H)$$

Substitute this identity in Eq. (30).

$$P \sinh At \pm S \cosh At = G \cosh H \sinh At \pm G \sinh H \cosh At$$

By inspection

$$P = G \cosh H$$

and

$$S = G \sinh H$$

$$\text{Now} \quad G^2 \cosh^2 H - G^2 \sinh^2 H = G^2 = P^2 - S^2$$

Therefore,  $G$  is evaluated in terms of  $P$  and  $S$ .

$$G = \sqrt{P^2 - S^2}$$

To find  $H$  in terms of  $P$  and  $S$

$$\tanh H = \frac{G \sinh H}{G \cosh H} = \frac{S}{P}$$

and

$$H = \tanh^{-1} \frac{S}{P}$$

Hence, the identity, Eq. (29), is proved.

A particular form of Eq. (29) is

$$\sinh At \cosh B \pm \cosh At \sinh B \equiv \sinh (At \pm B)$$

In this identity  $P = \cosh B$  and  $S = \sinh B$ , so

$$G = \sqrt{P^2 - S^2} = \sqrt{\cosh^2 B - \sinh^2 B} = 1$$

$$\text{and} \quad H = \tanh^{-1} \frac{\sinh B}{\cosh B} = \tanh^{-1} \tanh B = B$$

A trigonometric identity that is similar to Eq. (29) will also be used.

$$P \sin At \pm S \cos At \equiv \sqrt{P^2 + S^2} \sin \left( At \pm \tan^{-1} \frac{S}{P} \right)$$

The proof of this identity is analogous to that for the hyperbolic functions. It will be found that the following trigonometric identities are necessary for the proof:

$$\sin^2 At + \cos^2 At \equiv 1$$

$$\sin (At \pm B) \equiv \sin At \cos B \pm \cos At \sin B$$

$$\tan At \equiv \frac{\sin At}{\cos At}$$

**12. Imaginary Numbers.**—There are some interesting properties of the hyperbolic functions when the exponent is imaginary.

Before proceeding to show these properties it is necessary to know that

$$\epsilon^{\pm jAt} = \cos At \pm j \sin At^1$$

where  $j = \sqrt{-1}$ . Consider the hyperbolic sine of an imaginary number.

$$\sinh jAt = \frac{\epsilon^{jAt} - \epsilon^{-jAt}}{2}$$

Expand  $\epsilon^{jAt}$  and  $\epsilon^{-jAt}$ .

$$\sinh jAt = \frac{\cos At + j \sin At - \cos At + j \sin At}{2} = j \sin At$$

Another identity is

$$\cosh jAt \equiv \cos At$$

$$\begin{aligned} \text{Proof: } \cosh jAt &= \frac{\epsilon^{jAt} + \epsilon^{-jAt}}{2} \\ &= \frac{\cos At + j \sin At + \cos At - j \sin At}{2} = \cos At \end{aligned}$$

Thus it is seen that there is a connection between the trigonometric functions and the hyperbolic functions.

**13. Differentiation.**—The derivatives of the hyperbolic sine and hyperbolic cosine are of interest.

$$\frac{d}{dt} (\sinh At) = A \cosh At$$

$$\begin{aligned} \text{Proof: } \frac{d}{dt} (\sinh At) &= \frac{d}{dt} \left( \frac{\epsilon^{At} - \epsilon^{-At}}{2} \right) = \frac{1}{2} (A\epsilon^{At} + A\epsilon^{-At}) \\ &= A \left( \frac{\epsilon^{At} + \epsilon^{-At}}{2} \right) = A \cosh At \end{aligned}$$

$$\frac{d}{dt} (\cosh At) = A \sinh At$$

<sup>1</sup> To demonstrate that this equation is true, expand each of the three terms by Maclaurin's theorem.

$$\begin{aligned} \epsilon^{jAt} &= 1 + jAt - \frac{A^2 t^2}{2!} - j \frac{A^3 t^3}{3!} + \frac{A^4 t^4}{4!} + j \frac{A^5 t^5}{5!} + \cdots \\ \cos At &= 1 - \frac{A^2 t^2}{2!} + \frac{A^4 t^4}{4!} - \cdots \\ j \sin At &= +jAt - j \frac{A^3 t^3}{3!} + j \frac{A^5 t^5}{5!} + \cdots \end{aligned}$$

Therefore,  $\cos At + j \sin At \equiv \epsilon^{jAt}$   
and  $\epsilon^{-jAt} = \cos (-At) + j \sin (-At) = \cos At - j \sin At$

$$\begin{aligned}\text{Proof: } \frac{d}{dt} (\cosh At) &= \frac{d}{dt} \left( \frac{e^{At} + e^{-At}}{2} \right) = \frac{1}{2} (Ae^{At} - Ae^{-At}) \\ &= A \left( \frac{e^{At} - e^{-At}}{2} \right) = A \sinh At\end{aligned}$$

**14. Conclusion.**—The material on hyperbolic functions that has been presented in this chapter consists of miscellaneous fragments taken from the general subject.<sup>1</sup> This material represents the minimum amount that is required as background for the analytical work to follow.

The reason for using hyperbolic functions in this book is to place the mathematical results in a more concise form than is possible with the sole use of exponentials. Equations involving hyperbolic functions can be evaluated from the tables appearing in Appendix I.

### Problems

**Prob. 1.** Find the complete solution of the following differential equations:

(a)  $L \frac{di}{dt} + Ri = 0$

(b)  $R \frac{dq}{dt} + \frac{q}{C} = E$

(c)  $L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0; R > 2 \sqrt{\frac{L}{C}}$

**Prob. 2.** If  $R^2 = 4L/C$ , what is the solution of the following differential equation?

$$L \frac{di}{dt} + Ri + \frac{\int i dt}{C} = 0$$

**Prob. 3.** The current in a network is governed by the following differential equation:

$$Ri + \frac{\int i dt}{C} = E$$

- What is the steady-state value of current?
- What is the expression for the transient term?
- What is the ratio of the instantaneous current at  $t = RC$  to the instantaneous current at  $t = 2RC$ ?

**Prob. 4.** In Prob. 1b the instantaneous charge  $q$  is zero when  $t = 0$ .

- What is the steady-state value of charge?
- What is the rate of change of instantaneous charge at  $t = 0$ ?

<sup>1</sup> For a detailed treatment of hyperbolic functions refer to H. W. Reddick and F. H. Miller, "Advanced Mathematics for Engineers," Chap. II, John Wiley & Sons, Inc., New York, 1938.

**Prob. 5.** Find the value of  $At$  required to have

$$\sinh At + 0.1 = \cosh At$$

**Prob. 6.** Prove the identity

$$\sinh 2At \equiv 2 \sinh At \cosh At$$

**Prob. 7.** What restriction must be placed upon the relative values of  $A$  and  $B$  in the expression  $(e^{-Bt} \sinh At)$  to ensure that the expression has a limiting value of zero when  $t$  approaches infinity?

## CHAPTER III

### SERIES NETWORKS CONTAINING RESISTANCE AND CAPACITANCE

Series networks containing resistance and capacitance are relatively simple to analyze by means of ordinary differential equations, and the analysis yields very useful results. This type of network occurs repeatedly in many branches of electrical work, and a fundamental knowledge of its behavior is obtained by applying the classical method.

#### BASIC *RC* NETWORK WITH CAPACITANCE ACROSS OUTPUT

The four-terminal network shown in Fig. 12 is subjected to a rectangular-pulse voltage  $E$ . The object of this analysis is to find the resulting output voltage  $e$ .

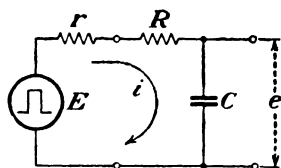


FIG. 12.—Basic series *RC* network with a rectangular-pulse generator. Capacitance output.

The differential-equation method requires two separate solutions: one for the time during which the generator pulse exists, and the other for the time after the generator pulse has disappeared.

**1. Equation for Output Pulse;  $t = 0$  to  $t = T$ .**—Considering the first interval, the differential equation that applies from  $t = 0$  to  $t = T$  is

$$E = ri_E + Ri_E + \frac{q_E}{C} \quad (31)$$

This is a statement of Kirchhoff's voltage law: the algebraic sum of all voltages around a closed network is zero. The subscript  $E$  signifies validity only during the time that the generator voltage is  $E$ . During the pulse interval the generator voltage has a constant value  $E$  that equals the sum of the instantaneous voltage across  $r$ ,  $ri_E$ ; the instantaneous voltage across  $R$ ,  $Ri_E$ ; and the instantaneous voltage across  $C$ ,  $q_E/C$ . Since the rate of change of charge  $dq_E/dt$  equals the instantaneous current  $i_E$ ,  $dq_E/dt$  can be inserted for  $i_E$  in Eq. (31).

$$E = (R + r) \frac{dq_E}{dt} + \frac{q_E}{C} \quad (31a)$$

This is a linear, first-order, first-degree equation that can be solved by the method of separation of variables. Separate  $t$  and  $q_E$ , the two variables.

$$\frac{dq_E}{E - (q_E/C)} = \frac{dt}{R + r}$$

Integrate both sides of the equation.

$$-C \int \frac{-(1/C) dq_E}{E - (q_E/C)} = \frac{1}{R + r} \int dt$$

The solution is

$$\ln \left( E - \frac{q_E}{C} \right) = - \frac{t}{(R + r)C} + K_1$$

The constant of integration  $K_1$  can be evaluated by investigating conditions at the time  $t = 0$ , the first instant at which the differential equation is applicable. At this time the generator voltage is  $E$ . The value of the instantaneous charge at  $t = 0$  can be found indirectly by rearranging Eq. (31a).

$$\frac{dq_E}{dt} = \frac{E - (q_E/C)}{R + r}$$

Inspection of this equation discloses that  $dq_E/dt$  is finite if  $(R + r)$  or  $C$  is not equal to zero, and if  $E$  is finite. A finite value of  $dq_E/dt$  means that it is impossible for charge to accumulate instantly. In other words, a finite time must elapse before charge can appear on  $C$ . Hence, at  $t = 0$ ,  $q_E = 0$ , and accordingly

$$K_1 = \ln E$$

Thus the solution becomes

$$\ln \left( E - \frac{q_E}{C} \right) = - \frac{t}{(R + r)C} + \ln E$$

which can be written

$$\ln \left[ \frac{E - (q_E/C)}{E} \right] = - \frac{t}{(R + r)C}$$

Convert to the exponential form.

$$\epsilon^{-\frac{t}{(R+r)C}} = \frac{E - (q_E/C)}{E}$$

Solve for  $q_E$ .

$$q_E = CE[1 - \epsilon^{-(R+r)C}] \quad (32)$$

The voltage at the output terminals is  $q_E/C$ . Hence, the solution for the output voltage during the pulse interval is

$$e_E = \frac{q_E}{C} = E[1 - \epsilon^{-\frac{t}{(R+r)C}}] \quad (33)$$

The internal generator resistance is included in the mathematical analysis because it is usually important and sometimes overlooked. However, Eq. (33) can be simplified if the generator resistance is very small compared with  $R$ , or mathematically if  $(r/R) \ll 1$ . Then the output voltage during the pulse interval simplifies to

$$e_E \approx E(1 - \epsilon^{-\frac{t}{RC}}) \quad (33a)$$

**2. Network Behavior;  $t = 0$  to  $t = T$ .**—With the analytical solution obtained, it is now appropriate to discuss the results descriptively and graphically and thereby to interpret Eqs. (32) and (33). During the time interval  $t = 0$  to  $t = T$ , the capacitor is charging from the pulse source, and the charge accumulates exponentially as illustrated in Fig. 13 and as indicated by Eq. (32).

The time equal to  $(R + r)C$  is defined as the *time constant*. At this time the instantaneous charge on  $C$  is 63.2 per cent of  $CE$ .<sup>1</sup>

$$(q_E)_{t=(R+r)C} = CE(1 - \epsilon^{-1}) = CE(1 - 0.368) = 0.632 CE$$

The time constant is useful for describing how quickly steady-state values are attained in capacitive or inductive circuits. The dimension of resistance times capacitance is time (Chap. I, page 19).

Another interpretation of the time constant is suggested in Fig. 13 by the sloping line through  $t = 0$ . This line is tangent to the growth-of-charge curve at  $t = 0$ , and its slope is therefore

<sup>1</sup> See Table of Exponentials, Appendix I.

equal to the initial rate of change of charge. If the initial rate of change of charge were maintained, the final value  $CE$  would be attained in a time equal to the time constant. This

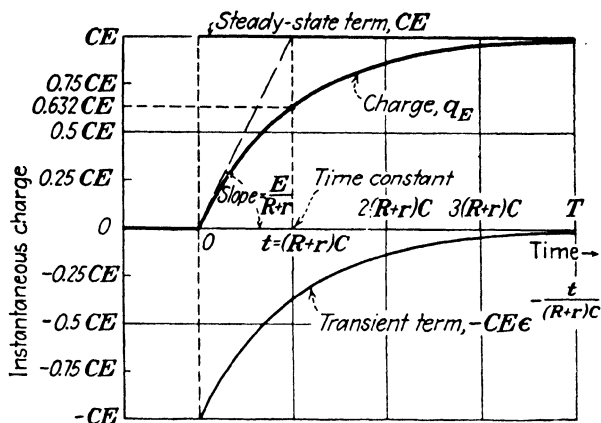


FIG. 13.—Exponential growth of charge on  $C$  for a pulse width equal to  $4(R+r)C$ . The sum of the transient and steady-state terms at any instant equals  $q_E$ .

can be demonstrated mathematically by setting  $q_E = 0$  in Eq. (31a) to evaluate the slope at  $t = 0$ .

$$\left(\frac{dq_E}{dt}\right)_{t=0} = \frac{E}{R+r}$$

If this rate of change of charge were maintained for a time equal to  $(R+r)C$ , then the value of charge at  $t = (R+r)C$  would be

$$(q_E)_{t=(R+r)C} = \frac{E}{R+r} (R+r)C = CE$$

Actually, the initial rate of change of charge is not maintained, and the charge on  $C$  approaches a value  $CE$  but reaches it only after infinite time. (This explains why it would not be useful to define a "time constant" that is equal to the time required for the steady state to be reached. Mathematically, such a "time constant" would have a value of infinity for most networks.) However, if the pulse duration is four times the time constant, the charge will be slightly more than 98 per cent of  $CE$ . To verify this, suppose that  $t = 4(R+r)C$ . From Eq. (32)

$$(q_E)_{t=4(R+r)C} = CE(1 - e^{-4}) = CE(1 - 0.0183) = 0.982 CE$$

In Fig. 13 two curves in addition to the growth-of-charge curve are shown. A study of these curves will cast further light on the network behavior during the interval  $t = 0$  to  $t = T$ . Equation (32) consists of two terms: one is  $CE$ , which is the value of charge approached, and the other is  $-CE\epsilon^{-\frac{t}{(R+r)C}}$ , which is a term becoming smaller as time increases. These two terms are very important. The first term,  $CE$ , which is constant, is the steady-state term because it is the value of charge on  $C$  after all network variations have died out. This term could have been obtained from the original differential equation by recognizing that the rate of change of charge  $dq_E/dt$  is zero when the steady state is reached. Placing  $dq_E/dt = 0$  in Eq. (31a) yields the steady-state term  $q_E = CE$ . Recall that the steady-state term is the same as the particular integral of the differential equation. The second term  $-CE\epsilon^{-\frac{t}{(R+r)C}}$  is the transient term because it contains all the information about the behavior of the varying conditions in the network. The transient term could have been found from the original differential equation by setting  $E = 0$  and solving for  $q_E$ . In other words, the transient term is the same as the complementary function. In Fig. 13 the transient term is largest at the instant the generator pulse appears, and then it decreases exponentially. Meanwhile, the sum of the steady-state and transient terms always equals the instantaneous charge. One viewpoint on the transient term is that it acts as a "shock absorber" that always makes up the difference between the existing value of charge and the steady-state value.

In most networks where transients occur the equations for voltage, current, or charge contain these two terms. It is useful to keep this idea clearly in mind whenever it becomes necessary to distinguish between voltage, current, and charge due to transient (time-varying) conditions or those due to steady-state (constant) conditions. For instance, if the pulse width is large compared with the time constant of the network, then at  $t = T$  the network transient is practically zero and the steady-state condition prevails. However, if the pulse width is comparable to the time constant, then in the interval  $t = 0$  to  $t = T$  the transient condition has not had time to become neg-

ligibly small, and it will continue to contribute to the network behavior from the time  $T$  on.

The output voltage will vary in the same manner as the charge since charge on  $C$  and voltage across  $C$  are directly proportional. Figure 14 shows the output voltage as a function of time during the pulse interval, Eq. (33), where the pulse interval is twice the time constant.

To compute the value of the output voltage at the instant the pulse disappears from the generator, Eq. (33) can be used.

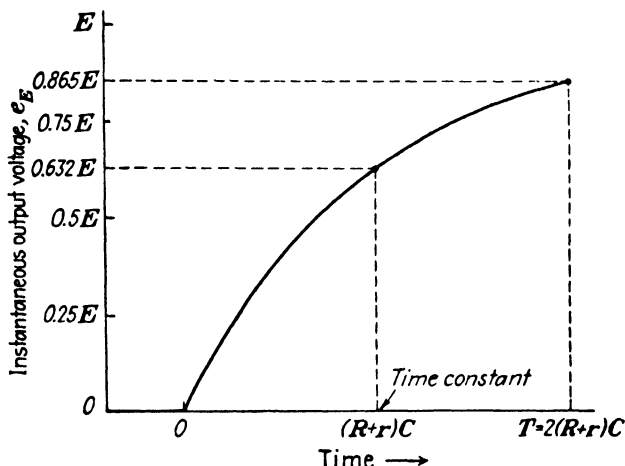


FIG. 14.—Output voltage of the network in Fig. 12 during the generator-pulse interval only.

For example, if the pulse width is twice the time constant, the output voltage at the instant the pulse disappears from the generator will be

$$e_{ET} = E(1 - e^{-2}) = E(1 - 0.135) = 0.865E$$

Of course Eq. (33) can be evaluated for any time  $t$ , and expressing  $t$  in terms of the time constant is done merely to simplify the calculations.

**3. Equation for Output Pulse;  $t = T$  to  $t = \infty$ .**—Equation (33) is only part of the solution because the pulse will disappear from the generator terminals after a time  $T$ , and the output voltage will no longer behave in the manner indicated by Eq. (33) and Fig. 14. To obtain the rest of the solution, the differential equation that applies at the instant the pulse disappears from the

generator must be considered. The required equation is obtained by setting  $E$  equal to zero in Eq. (31).

$$0 = (R + r)i_0 + \frac{q_0}{C}$$

The subscript 0 signifies validity only during the time that the generator voltage is zero. Substitute for  $i_0$  its equal,  $dq_0/dt$ .

$$(R + r) \frac{dq_0}{dt} = - \frac{q_0}{C}$$

Separate variables and integrate.

$$\int \frac{dq_0}{q_0} = - \frac{1}{(R + r)C} \int dt$$

The solution is

$$\ln q_0 = - \frac{t}{(R + r)C} + K_2 \quad (34)$$

To evaluate the constant of integration  $K_2$ , the instantaneous charge on  $C$  at the time the pulse disappears from the generator must be known. From Eq. (32) its value is

$$q_{ET} = CE[1 - e^{-\frac{T}{(R+r)C}}]$$

where  $q_{ET}$  denotes the instantaneous charge on  $C$  at the time  $T$ .  $K_2$  is accordingly

$$K_2 = \ln \{CE[1 - e^{-\frac{T}{(R+r)C}}]\} + \frac{T}{(R + r)C}$$

Therefore, Eq. (34) becomes

$$\ln q_0 = - \frac{t}{(R + r)C} + \ln \{CE[1 - e^{-\frac{T}{(R+r)C}}]\} + \frac{T}{(R + r)C}$$

which can be written

$$\ln \left\{ \frac{q_0}{CE[1 - e^{-\frac{T}{(R+r)C}}]} \right\} = - \frac{(t - T)}{(R + r)C}$$

Convert to the exponential form.

$$q_0 = CE[1 - e^{-\frac{T}{(R+r)C}}]e^{-\frac{(t-T)}{(R+r)C}}$$

This simplifies to

$$q_0 = CE[\epsilon^{\frac{T}{(R+r)C}} - 1]\epsilon^{-\frac{t}{(R+r)C}} \quad (35)$$

The output voltage is

$$e_0 = \frac{q_0}{C} = E[\epsilon^{\frac{T}{(R+r)C}} - 1]\epsilon^{-\frac{t}{(R+r)C}} \quad (36)$$

Thus the complete solution for the applied rectangular pulse is twofold. During the generator pulse the output voltage is governed by Eq. (33), and after the generator pulse has disappeared, the output voltage varies in accordance with Eq. (36).

The form of Eq. (36), a constant times an exponential, is fixed. Under special conditions, however, Eq. (36) takes on a simpler appearance. Suppose, for instance, that  $r$  is negligible compared with  $R$ . Then Eq. (36) could be written

$$e_0 \approx E(\epsilon^{\frac{T}{RC}} - 1)\epsilon^{-\frac{t}{RC}} \quad (36a)$$

If the pulse width were large compared with the time constant, then  $e_E$  would be practically equal to  $E$  when the pulse disappears, and Eq. (36a) could be further simplified to

$$e_0 \approx E\epsilon^{-\frac{(t-T)}{RC}} \quad (36b)$$

Mathematically, this can be shown by recognizing that if  $RC \ll T$ , then

$$\epsilon^{\frac{T}{RC}} - 1 \approx \epsilon^{\frac{T}{RC}}$$

**4. Network Behavior;  $t = T$  to  $t = \infty$ .**—Equations (35) and (36) will now be discussed in terms of the network. At the instant the pulse disappears from the generator, the capacitor stops charging and commences discharging through  $R$  and the generator resistance  $r$ . The charge decreases in accordance with Eq. (35). Suppose the pulse width is large compared with  $(R + r)C$ . Then when the generator pulse disappears at  $t = T$ , the charge on  $C$  will be practically  $CE$ . The charge will decay exponentially as indicated in Fig. 15. After a discharge time equal to  $(R + r)C$ , the charge on  $C$  will have been reduced by 63.2 per cent or will be 36.8 per cent of  $CE$ . This can be deduced by recognizing that Eq. (35) becomes  $q_0 = CE\epsilon^{-\frac{(t-T)}{(R+r)C}}$

if  $T \gg (R+r)C$ . Now, when  $t = T + (R+r)C$ , then  $q_0 = CE\epsilon^{-1} = 0.368 CE$ . Mathematically, the charge reaches zero only after an infinite interval of time, but the remaining

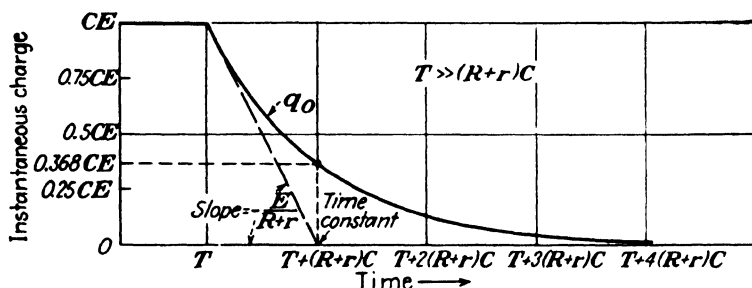


FIG. 15.—Exponential decay of charge on  $C$  for a pulse width large compared with  $(R+r)C$ . The value of charge at  $t = T$  equals the steady-state value of charge shown in Fig. 13.

$$T_1 = 0.4(R+r)C \quad T_2 = 1.6(R+r)C \quad T_3 = 4.8(R+r)C \quad T_4 = 10(R+r)C$$

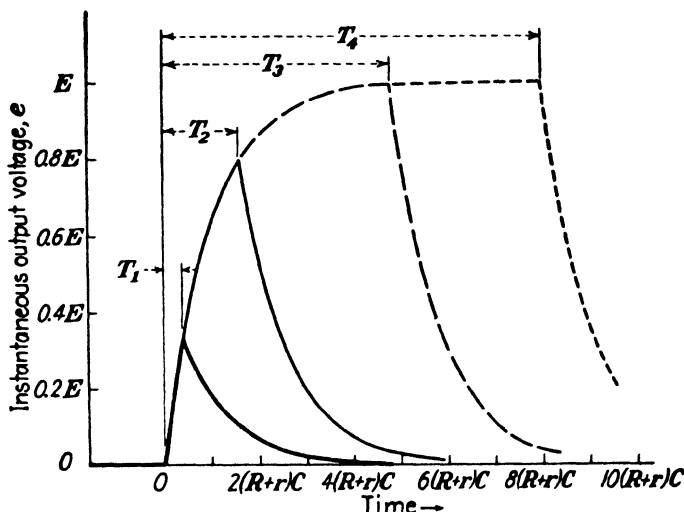


FIG. 16.—Complete output voltage of the network in Fig. 12 for four generator pulses of equal amplitude  $E$  but different pulse widths.

charge is less than 2 per cent of  $q_{ET}$  after a time equal to four times the time constant. This is true irrespective of the value of charge at  $t = T$ . To demonstrate this, substitute

$$t = T + 4(R+r)C$$

into Eq. (35).

$$\begin{aligned}
 q_0 &= CE[\epsilon^{\frac{T}{(R+r)C}} - 1]\epsilon^{-\frac{T+4(R+r)C}{(R+r)C}} + CE[\epsilon^{\frac{T}{(R+r)C}} - 1]\epsilon^{-\frac{T}{(R+r)C}}\epsilon^{-4} \\
 &= CE[1 - \epsilon^{-\frac{T}{(R+r)C}}]\epsilon^{-4} = 0.0183q_{ET}
 \end{aligned}$$

The steady-state term in eq. (35) is zero because the voltage applied to the network from the time  $T$  on is zero. Consequently, the transient term accounts for all of the charge in the network. After the transient has become negligible, the steady-state condition of zero charge prevails.

Figure 16 reveals the complete history of output voltage versus time for generator pulses of various time durations and equal amplitudes. The oscillogram in Fig. 17 is an illustration of the output voltage obtained in a practical  $RC$  network. The charging time constant is less than the discharging time constant because the internal generator resistance is smaller during the generator pulse than after the generator pulse.



FIG. 17.—Output pulse of the network in Fig. 12 for a pulse width that is large compared with the network time constant.

**5. General Network Behavior.**—Some general remarks concerning the behavior of the network are pertinent now. Figure 16 indicates that the output voltage approaches more and more closely an exact reproduction of the input voltage as the time constant is made smaller compared with the pulse width. This can be qualitatively explained on the basis of the definition of the time constant, which states that the time constant is directly proportional to both the resistance and the capacitance. There are essentially two means by which the time constant can be decreased: reduction of resistance, or reduction of capacitance. If the resistance is reduced, the amount of charge required to produce a given output voltage can accumulate more rapidly, since there is less resistance to the flow of charge. If the capacitance is reduced, the amount of charge required to produce the same output voltage is less. In either case, the output voltage tends to follow the generator pulse more closely.

It also can be proved mathematically that the output pulse will approach the shape of the generator pulse when the network time constant is small compared with the pulse width. Furthermore, this is true irrespective of the generator-pulse

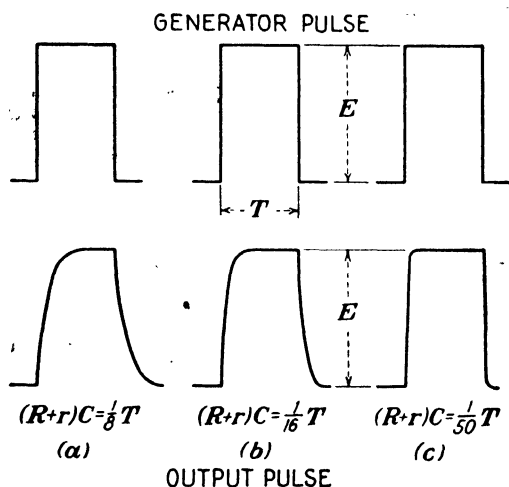


FIG. 18.—The output pulse duplicates the generator pulse more exactly in shape and amplitude as the network time constant is made smaller compared with the pulse width.

shape. If any generator voltage  $e_g$  is applied to the network in Fig. 12, the equation for instantaneous voltage is

$$e_g = (R + r)i + \frac{q}{C}$$

which can be written

$$Ce_g = i(R + r)C + q$$

The term  $i(R + r)C$  becomes small compared with  $q$  almost immediately because (1) the steady-state value of charge is attained almost immediately when  $(R + r)C$  is small, and (2) the instantaneous current becomes small as the steady-state value of charge is approached. Therefore,

$$Ce_g \approx q$$

and

$$e_g \approx \frac{q}{C} = e; \quad (R + r)C \ll T$$

This shows that the output voltage  $e$  is approximately equal to the generator voltage. In Fig. 18 the generator and output voltages are illustrated for networks that have time constants equal to  $\frac{1}{8}$ ,  $\frac{1}{16}$ , and  $\frac{1}{50}$  of the pulse width.

Conversely, as  $(R + r)C$  is made larger compared with the pulse width, the output voltage becomes less similar to the generator pulse voltage in both shape and amplitude. This behavior can be readily understood qualitatively: if the voltage builds up too slowly across  $C$ , the generator pulse will have

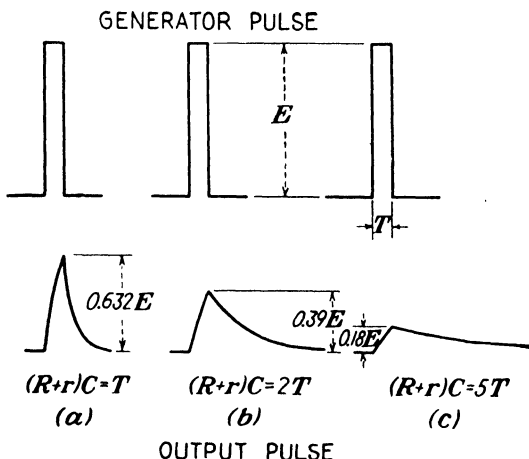


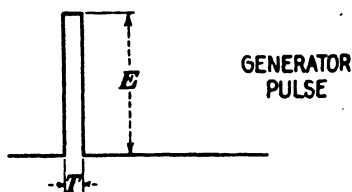
FIG. 19.—The output pulse departs more and more from the generator pulse in both shape and amplitude as the network time constant is made larger compared with the pulse width.

disappeared before the voltage across  $C$  has reached the pulse voltage  $E$ . Figure 19 presents output voltages for networks of time constants equal to 1.0, 2.0, and 5.0 times the pulse duration.

**6. Pulse Integration.**—Figure 19c suggests a use for the network in Fig. 12 because the output pulse is approximately the integral of the input pulse. As a matter of fact, this network is used to “integrate” pulses, not only rectangular in shape but of various shapes. The property of integration can be verified in a crude way by graphically taking the derivative (slope) of the output voltage wave form and observing how closely it resembles the input voltage. (This is valid because if a second quantity is the integral of the first quantity, then the derivative

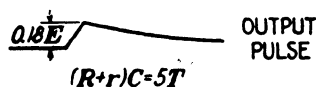
of the second quantity is the first.) This has been done in Fig. 20 where the property of integration is crudely indicated. The oscillogram in Fig. 21 shows an output voltage that approaches the integral of the generator pulse even more closely than does Fig. 19c.

Another more general means of verifying the property of integration is to apply to this network any voltage  $e_g$ , a function



of time only. The equation that sums all of the instantaneous voltages around the network is

$$e_g = (R + r)i + \frac{q}{C}$$



which can be written

$$\frac{e_g}{R + r} = i + \frac{q}{(R + r)C}$$



FIG. 20.—The output pulse is approximately the integral of the generator pulse when

$$(R + r)C \gg T$$

The lowest curve is obtained from the slope of the output pulse. Before the interval  $T$ , the slope of the output pulse is zero. During the interval  $T$ , the slope of the output pulse is positive and essentially constant. After the interval  $T$ , the slope of the output pulse is negative but smaller in magnitude than the slope in the interval  $T$ .

The term  $q/(R + r)C$  is negligible compared with  $i$  when  $(R + r)C$  is large compared with the generator-pulse width. This is because a very small amount of charge accumulates on  $C$ , and the current  $i$  is very nearly equal to its maximum value during the entire generator pulse. In other words, almost all of the generator voltage appears across  $(R + r)$  during the generator pulse, and the current is mainly determined by  $(R + r)$  and  $e_g$ . The resulting equation is

$$\frac{e_g}{R + r} \approx i = \frac{dq}{dt}$$

which, when solved for  $q$ , yields

$$q \approx \frac{1}{R + r} \int e_g dt$$

Hence, the output voltage is

$$e = \frac{q}{C} \approx \frac{1}{(R+r)C} \int e_s dt; \quad (R+r)C \gg T$$

and is approximately proportional to the integral of the input voltage, irrespective of its shape.

#### BASIC RC NETWORK WITH RESISTANCE ACROSS OUTPUT

If the series network is rearranged as in Fig. 22 so that the output voltage appears across  $R$  instead of  $C$ , the network will exhibit characteristics that are quite different from the case already considered. This network is also encountered frequently. The mathematical solution has in effect already been obtained and needs only slight reworking for application to this particular network. The fundamental point to recognize is that the time derivative of the instantaneous charge, equations for which have already been derived, is the instantaneous current. With the instantaneous current in the network known, the instantaneous output voltage is easily found.

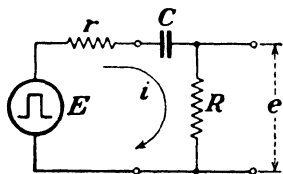


FIG. 22.—Basic series RC network with a rectangular pulse generator. Resistance output.

**7. Equation for Output Pulse;  $t = 0$  to  $t = T$ .**—Taking the time derivative of Eq. (32), which applies for the time interval during which the generator pulse exists, the equation for the instantaneous current is

$$i_E = \frac{dq_E}{dt} = \frac{E}{R+r} \epsilon^{-\frac{t}{(R+r)C}} \quad (37)$$

The instantaneous output voltage, therefore, is

$$e_E = Ri_E = \frac{ER}{R+r} \epsilon^{-\frac{t}{(R+r)C}} \quad (38)$$

If the generator resistance is small compared with  $R$ , i.e.,  $(r/R) \ll 1$ , then it is reasonable to approximate the output

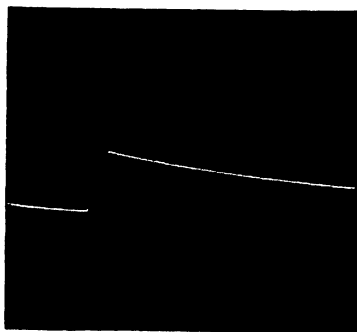


FIG. 21.—Oscillogram of the output voltage observed in an "integrating" network.

voltage during the pulse interval by the equation

$$e_g \approx E e^{-\frac{t}{RC}} \quad (38a)$$

**8. Network Behavior;  $t = 0$  to  $t = T$ .**—Equation (38) can be interpreted in terms of the current in the network. At the instant the generator pulse arrives, the charge on  $C$  is zero, since it takes a finite time for charge to accumulate. Therefore, at this instant the voltage across  $C$  must be zero. Nevertheless, the generator voltage is  $E$  at  $t = 0$  and an equal voltage must be developed somewhere else in the network. The voltage

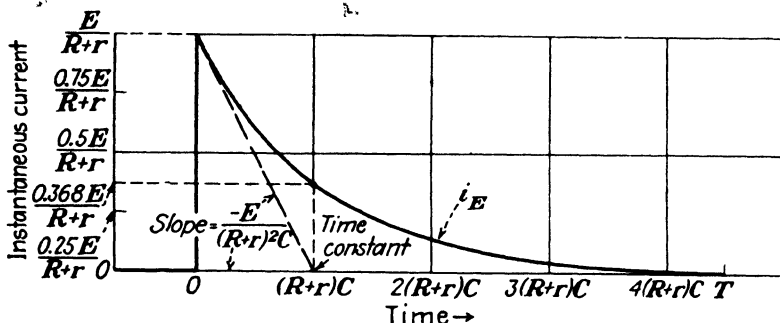


FIG. 23.—Exponential decay of current for a pulse width large compared with  $(R + r)C$ . The steady-state value of current is zero.

appears across  $R$  and  $r$ , and must be exactly equal to  $E$ . In order to produce such a voltage, current must flow from the generator at  $t = 0$  and must have a value equal to  $E/(R + r)$ . Equation (37) contains this information, for if  $t = 0$ , then  $i_g = E/(R + r)$ . In other words, at  $t = 0$  the instantaneous charge on the capacitor is zero, but the time rate of change of charge, which is the instantaneous current, is not zero. Refer to Fig. 13 where the slope of the growth-of-charge curve is equal to the instantaneous current. When current flows, charge must accumulate on  $C$ , and consequently a voltage proportional to this charge will appear across  $C$ . This means that less current will flow from the generator because the voltage across  $(R + r)$  will be  $E$  minus the voltage across  $C$ . Thus the current decreases in the same manner in which the charge on  $C$  increases, namely, exponentially. This also explains why the initial rate of change of charge is not maintained. Figure 23 shows the behavior

of the current as indicated by Eq. (37) up to the time that the generator pulse disappears. If this curve is turned over in the time interval from  $t = 0$  on, it will conform to the curve for growth of charge on  $C$ . The same time-constant considerations apply; consequently, at  $t = (R + r)C$ ,  $i_E = 0.368E/(R + r)$ , and at  $t = 4(R + r)C$  the current will be less than 2 per cent of  $E/(R + r)$ .

The output voltage will vary in the same manner as the current since the output voltage is directly proportional to the current. Figure 24 illustrates the output voltage as a func-

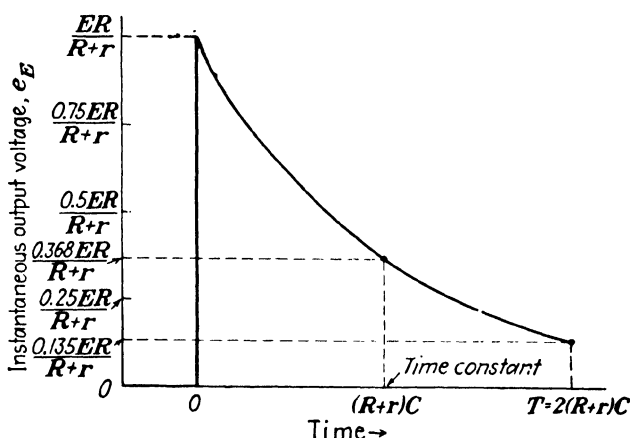


FIG. 24.—Output voltage of the network in Fig. 22 during the generator-pulse interval only.

tion of time during the generator-pulse interval when the pulse interval is twice the time constant.

To compute the positive output voltage at the instant the generator pulse falls to zero voltage,  $t = T$  can be inserted into Eq. (38). For example, suppose the pulse width is three times the time constant. Then from Eq. (38)

$$(e_E)_{t=3(R+r)C} = \frac{ER}{R+r} e^{-3} = 0.0498 \frac{ER}{R+r}$$

**9. Equation for Output Pulse;  $t = T$  to  $t = \infty$ .**—To obtain the solution for the time after the pulse disappears, Eq. (35) can be used. Taking the time derivative of Eq. (35), the equation for instantaneous current after the pulse disappears is

$$i_0 = \frac{dq_0}{dt} = -\frac{E}{R+r} [\epsilon^{\frac{T}{(R+r)C}} - 1] \epsilon^{-\frac{t}{(R+r)C}} \quad (39)$$

The instantaneous output voltage is then

$$e_0 = Ri_0 = -\frac{ER}{R+r} [\epsilon^{\frac{T}{(R+r)C}} - 1] \epsilon^{-\frac{t}{(R+r)C}} \quad (40)$$

The output voltage is zero before the time  $t = 0$ , Eq. (38) describes the output voltage from  $t = 0$  to  $t = T$ , and Eq. (40) expresses the output voltage from  $t = T$  to  $t = \infty$ . Thus the complete output voltage behavior has been found.

If the generator resistance is negligible compared with  $R$ , then Eq. (40) simplifies to

$$e_0 \approx -E(\epsilon^{\frac{T}{RC}} - 1) \epsilon^{-\frac{t}{RC}} \quad (40a)$$

This equation takes on a simpler appearance if the additional condition that the time constant is much less than the pulse width is applicable. It then becomes

$$e_0 \approx -E\epsilon^{-\frac{(t-T)}{RC}} \quad (40b)$$

because  $\epsilon^{\frac{T}{RC}} - 1 \approx \epsilon^{\frac{T}{RC}}$  when  $RC \ll T$ .

**10. Network Behavior;  $t = T$  to  $t = \infty$ .**—At the instant the generator pulse disappears,  $t = T$ , and current stops flowing from the generator, but at that instant  $(R + r)$  is subjected to the voltage attained across  $C$  during the pulse. Hence, current will flow, but now in the opposite direction, since formerly  $C$  was charging and now  $C$  is discharging. Therefore, there will be a sharp discontinuity in current flow at the instant the pulse disappears from the generator. As time goes on, the charge on  $C$  decreases because it is discharging through  $r$  and  $R$  in series; consequently, the magnitude of the current decreases. Equation (39), which applies to the discharge interval after the generator pulse disappears, is drawn in Fig. 25. The current is shown negative, which signifies that its direction has been reversed. The output voltage is directly proportional to  $i_0$ , and hence its polarity will also be reversed.

**11. Network Behavior;  $t = 0$  to  $t = \infty$ .**—The graphical picture of output voltage for all times can now be considered in



negative value of  $e_0$  for any pulse width. The envelope is shown in Fig. 26 where it is clearly seen that it consists of the positive output-voltage curve displaced in the negative direction by an amount  $ER/(R+r)$ . The oscillogram in Fig. 27 pictures the output voltage across the resistance when the generator-pulse width is slightly less than the network time constant.

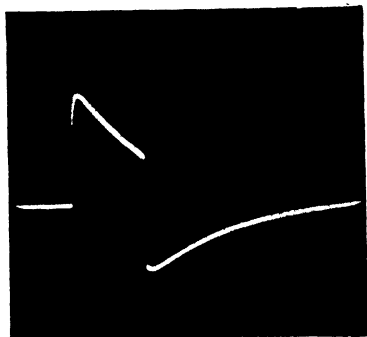


FIG. 27.—Output pulse of the network in Fig. 22 when the pulse width is slightly less than the network time constant.

**12. General Network Behavior.**—The behavior of this network is quite opposite from the behavior of the network where the output voltage is taken across  $C$ . Recall that in the

latter case, as the network time constant was made smaller compared with the pulse width, the output voltage became more closely similar in shape to the input pulse. However, in this network the opposite is true: as the network time constant is made

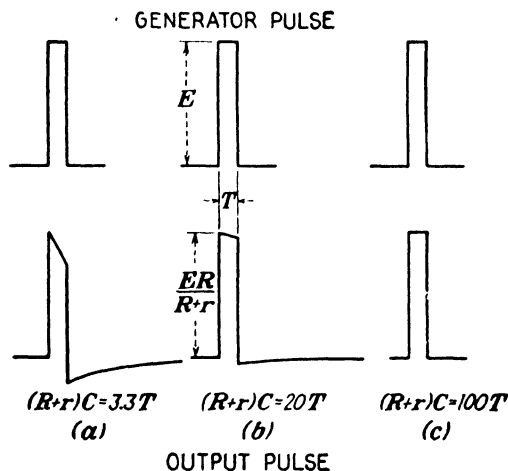


FIG. 28.—The output-pulse shape duplicates the generator-pulse shape more closely as the network time constant is made larger compared with the pulse width.

smaller compared with the pulse width, the output voltage differs more in shape from the input pulse; and when the time

constant is made larger, the output pulse approaches the shape of the input pulse. To explain this qualitatively consider the following: If the time constant is large, charge will accumulate very slowly on  $C$ . Now the appearance of voltage across  $C$  is the only thing in the network that will distort the output-pulse shape. (If the voltage on  $C$  is always zero, then the output voltage will have the exact shape of the generator pulse.) It is reasonable, then, to say that the output pulse will resemble the input pulse more closely if charge cannot quickly flow on  $C$ . Conversely, if charge *can* accumulate on  $C$  quickly compared with the time duration of the pulse, then the presence of voltage across  $C$  will introduce a different shape of output pulse. These remarks are illustrated in Figs. 28 and 29.

The fact that the output pulse resembles the input pulse more closely as the network time constant is made large compared with the generator-pulse width can be verified mathematically. For the network in Fig. 22 and for any generator voltage  $e_g$ ,

$$e_g = (R + r)i + \frac{q}{C}$$

or

$$Ce_g = i(R + r)C + q$$

Now  $q$  is negligible compared with  $i(R + r)C$  when  $(R + r)C$  is very large compared with the generator-pulse width. Therefore,

$$e_g \approx i(R + r)$$

or

$$\frac{e_g R}{R + r} \approx iR$$

However,  $e = iR$ , and consequently

$$e \approx \frac{R}{R + r} e_g; \quad (R + r)C \gg T$$

This demonstrates that the output voltage is approximately proportional to the generator voltage, irrespective of its shape.

**13. Pulse Differentiation.**—It has been shown in the network where the output voltage is taken across  $C$  that it is possible for the output pulse to be approximately the integral of the generator pulse (see pages 51–53). Thus far an opposite behavior has been seen for the same network but with voltage taken across  $R$ . It is not surprising to find that the opposite of integration, which is differentiation, can be performed by this network. A

careful examination of Fig. 29 will show that as the time constant is made smaller compared with the pulse width, the output wave

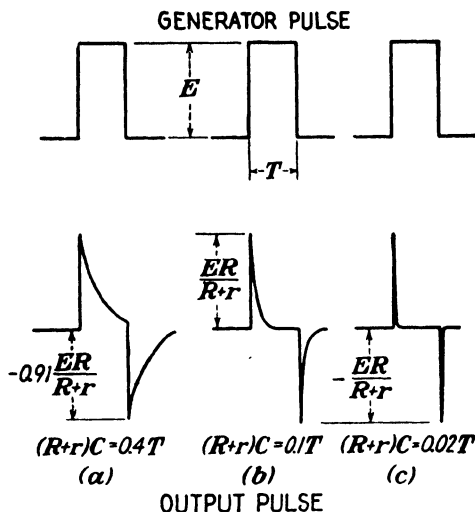


FIG. 29.—The output-pulse shape departs more and more from the generator-pulse shape as the network time constant is made smaller compared with the pulse width. If the network time constant is made sufficiently small, the output voltage is approximately the derivative of the generator pulse.

form approaches the derivative of the generator pulse. When an extremely small time constant is approached, Fig. 29c, the output pulse will consist of two lines, which is approximately the derivative (slope) of the generator pulse. Figure 30 illustrates the output voltage of a practical "differentiating" network.

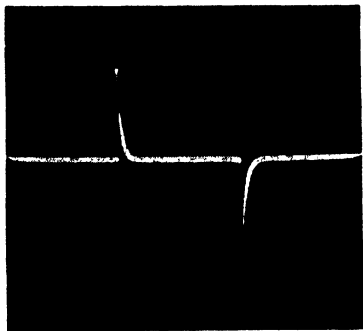


FIG. 30.—Oscillogram of the output voltage observed in a "differentiating" network.

all of the instantaneous voltages around the network is

$$e_o = (R + r)i + \frac{q}{C}$$

which can be written

$$Ce_g = i(R + r)C + q$$

When  $(R + r)C$  is very small compared with the generator-pulse width, then  $i(R + r)C$  is negligible compared with  $q$  and  $Ce_g \approx q$ . Take the derivative.

$$C \frac{de_g}{dt} \approx \frac{dq}{dt}$$

However,  $i = dq/dt$  and hence  $i \approx C (de_g/dt)$ . The output voltage will be

$$e = iR \approx RC \frac{de_g}{dt}; \quad (R + r)C \ll T$$

and is approximately proportional to the derivative of the generator voltage, irrespective of its shape.

#### POWER AND ENERGY RELATIONS

Before proceeding with the analysis of other series networks containing resistance and capacitance, it is interesting to investigate the power and energy relations for the two simple networks discussed thus far.

**14. Energy Transfer and Transformation.**—The generator delivers current to the network during the time interval  $t = 0$  to  $t = T$ . This current flow represents a transfer of energy from the generator to the network. A portion of this transferred energy is dissipated in the resistance and transformed into heat; the remainder is stored in the electric field of the capacitor. From the time  $t = T$  on, the generator no longer supplies energy. In general, however, current continues to flow after the time  $t = T$ . The source of this current is the stored energy in the electric field of the capacitor. All the stored energy is eventually dissipated as heat in the resistance. Thus the qualitative energy picture is clear.

**15. Instantaneous Power and Instantaneous Total Energy.**—If more than a qualitative idea of the energy transfer and transformation is desired, it is necessary to perform a mathematical analysis. This can be done in essentially two steps: (1) power analysis, and (2) energy analysis. Before the power and energy analyses can be made, mathematical definitions of electric power and energy are necessary.

TABLE II.—POWER AND ENERGY RELATIONS IN A BASIC RC NETWORK

	Time interval $t = 0$ to $t = T$	Time interval $t = T$ to $t = \infty$
Generator voltage, $e_g$	$e_g = E$	$e_g = 0$
Voltage across $R + r$ , $e_{R+r}$	$e_{R+r} = (R + r)i_E$	$e_{R+r} = (R + r)i_0$
Voltage across $C$ , $e_C$	$e_C = \frac{q_E}{C}$	$e_C = \frac{q_0}{C}$
Generator power, $p_g$	$p_g = Ei_E$	$p_g = 0$
Resistance power, $p_{R+r}$	$p_{R+r} = (R + r)i_E^2$	$p_{R+r} = (R + r)i_0^2$
Capacitance power, $p_C$	$p_C = \frac{q_E}{C} i_E$	$p_C = \frac{q_0}{C} i_0$
Supplied energy (from generator), $w_g$	$w_g = \int_0^T Ei_E dt$	$w_g = 0$
Dissipated energy (in resistance), $w_{R+r}$	$w_{R+r} = \int_0^T (R + r)i_E^2 dt$	$w_{R+r} = \int_T^\infty (R + r)i_0^2 dt$
Stored energy (in capacitance), $w_C$	$w_C = \int_0^T \frac{q_E}{C} i_E dt$	$w_C = \int_T^\infty \frac{q_0}{C} i_0 dt$
Instantaneous charge, $q$	$q_E = CE[1 - e^{-\frac{t}{(R+r)C}}]$	$q_0 = CE[e^{\frac{T}{(R+r)C}} - 1]e^{-\frac{t}{(R+r)C}}$
Instantaneous current, $i$	$i_E = \frac{E}{R + r} e^{-\frac{t}{(R+r)C}}$	$i_0 = -\frac{E}{R + r} [e^{\frac{T}{(R+r)C}} - 1]e^{-\frac{t}{(R+r)C}}$

Power is defined as the product of voltage and current. In the networks under consideration all voltages and currents may be functions of time, and consequently their products, power, will also be functions of time. Therefore, power, as well as current and voltage, is an instantaneous quantity since it generally will have a different value at each instant. Equation (41) relates instantaneous power,  $p_1$ , for any network element to the instantaneous current flowing through that element,  $i_1$ , and the instantaneous voltage produced across that element,  $e_1$ .

$$p_1 = e_1 i_1 \quad (41)$$

Mathematically, energy is the time integral of power and is defined by Eq. (42).

$$w_1 = \int e_1 i_1 dt \quad (42)$$

This integral will have a value that depends upon the time interval over which the integration is performed, so the energy depends upon the time interval considered. The energy is not the same sort of instantaneous function as current, voltage, and power. The relation between current, voltage, power, and time gives the existing instantaneous power at any specified time. But the relation between energy and time gives the total or sum of the energy up to any specified time. In other words, it is the total energy stored or dissipated over a time interval that starts and ends at specified times. When this time interval is measured from  $t = 0$ , the energy will be called the *instantaneous total energy*. It is important to understand clearly this definition of instantaneous total energy when an interpretation is made of the mathematical results.

Table II lists the fundamental equations and the special forms of the power and energy equations for the network elements to be considered.

**16. Power Relations.**—First the power relations will be discussed. Direct substitution of the equations for  $q$  and  $i$  into the power equations yields

$$\begin{aligned} \text{DURING PULSE} \\ p_0 &= \frac{E^2}{R+r} e^{-\frac{t}{(R+r)C}} \\ p_{R+r} &= \frac{E^2}{R+r} e^{-\frac{2t}{(R+r)C}} \end{aligned}$$

$$p_c = \frac{E^2}{R+r} [\epsilon^{-\frac{t}{(R+r)C}} - \epsilon^{-\frac{2t}{(R+r)C}}]$$

AFTER PULSE

$$p_g = 0$$

$$p_{R+r} = \frac{E^2}{R+r} [\epsilon^{\frac{T}{(R+r)C}} - 1]^2 \epsilon^{-\frac{2t}{(R+r)C}}$$

$$p_c = -\frac{E^2}{R+r} [\epsilon^{\frac{T}{(R+r)C}} - 1]^2 \epsilon^{-\frac{2t}{(R+r)C}}$$

These equations are represented graphically in Fig. 31 for a pulse width that is equal to four times the time constant. Curves for  $i$  and  $q$  are also indicated. They suggest graphically the means by which the power curves can be obtained.

*Power during Pulse.*—Power is delivered by the generator only during the time interval  $t = 0$  to  $t = T$ . Refer to Fig. 31a. In this interval the instantaneous supplied power decreases exponentially with time in accordance with a time constant equal to  $(R+r)C$ . This power is delivered to the resistance and the capacitance. The instantaneous power being delivered to  $(R+r)$ , Fig. 31b, decreases exponentially with time in accordance with a time constant equal to  $\frac{1}{2}(R+r)C$ . In the early part of the interval  $t = 0$  to  $t = T$ , the instantaneous power delivered to  $(R+r)$  is greater than that delivered to  $C$ , which is evident from a comparison of Figs. 31b and 31c. The instantaneous power delivered to  $(R+r)$  is a maximum at  $t = 0$ , while the instantaneous power delivered to  $C$  reaches a maximum during the pulse interval at a time equal to  $0.693(R+r)C$ , the maximum value being  $\frac{1}{4}E^2/(R+r)$ .<sup>1</sup> If the pulse width is made small compared with the time constant, it can be deduced that practically all of the generator power will be delivered to the resistance and practically none to the capacitance. Thus, when the net-

<sup>1</sup> To obtain this maximum value, take the time derivative of  $p_c$  and equate to zero. Substitute the resulting value into the equation for  $p_c$ .

$$\frac{dp_c}{dt} = \frac{d}{dt} \left\{ \frac{E^2}{R+r} [\epsilon^{-\frac{t}{(R+r)C}} - \epsilon^{-\frac{2t}{(R+r)C}}] \right\} = 0$$

$$\epsilon^{-\frac{t}{(R+r)C}} = \frac{1}{2}; \quad t = 0.693(R+r)C$$

Therefore, 
$$p_{c_{\max}} = \frac{E^2}{R+r} \left( \frac{1}{2} - \frac{1}{4} \right) = \frac{1}{4} \frac{E^2}{R+r}$$

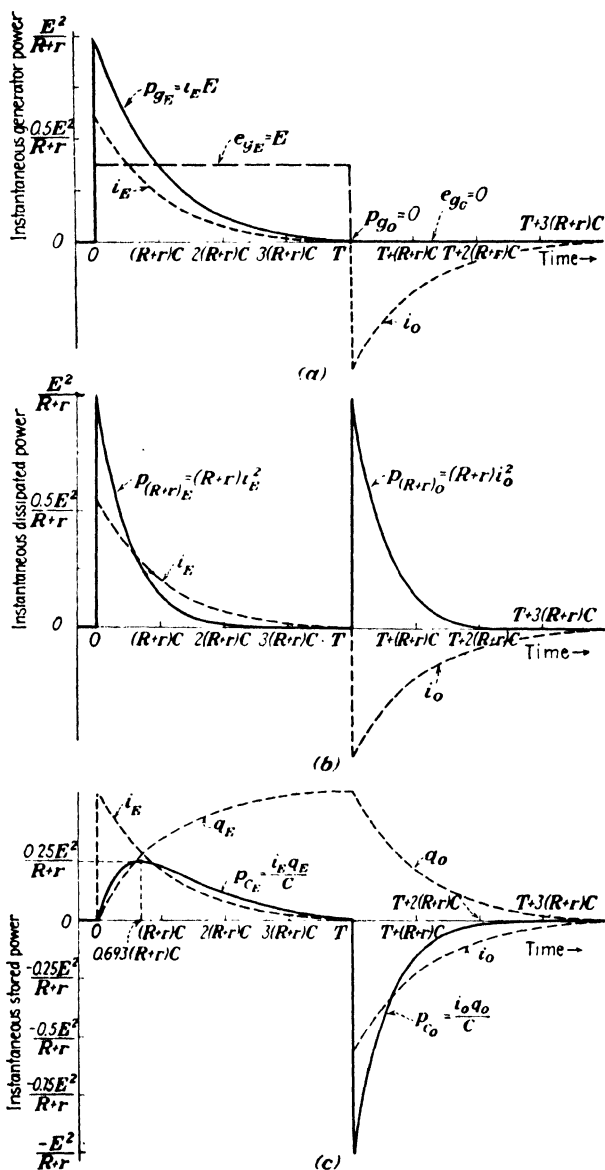


FIG. 31.—Behavior of instantaneous power in a basic  $RC$  network that is subjected to a rectangular-pulse voltage. The pulse width is equal to  $4(R+r)C$ .

work is used for integration, very little power is delivered to the output terminals.

The equations for instantaneous power during the pulse interval show that the sum of the instantaneous power delivered to  $(R + r)$  and to  $C$  is always equal to the instantaneous power supplied by the generator; *i.e.*,

$$p_g \equiv p_{R+r} + p_c$$

*Power after Pulse.*—In the time interval from  $t = T$  on, the power previously delivered to  $C$  is released and delivered to  $(R + r)$ . A glance at the power equations for this time interval reveals that  $p_c$  and  $p_{R+r}$  are always equal and opposite. The significance of negative instantaneous power is that the direction of current flow is opposite to the polarity of the voltage. The power delivered by  $C$  to  $(R + r)$  decreases exponentially in accordance with a time constant equal to  $\frac{1}{2}(R + r)C$ .

**17. Energy Relations.**—After this treatment of instantaneous power relations, it is possible to proceed to the instantaneous total energy relations. As indicated in Table II, integration of the power equations is necessary to obtain the energy equations. This integration is straightforward and has been carried out with the following results:

#### DURING PULSE

$$w_g = CE^2[1 - e^{-(R+r)Ct}]$$

$$w_{R+r} = \frac{1}{2}CE^2[1 - e^{-(R+r)Ct}] \quad (43)$$

$$w_c = \frac{1}{2}CE^2[1 + e^{-(R+r)Ct} - 2e^{-(R+r)Ct}] \quad (44)$$

#### AFTER PULSE

$$w_g = 0$$

$$w_{R+r} = \frac{1}{2}CE^2\{2[1 - e^{-(R+r)CT}] - [e^{-(R+r)CT} - 1]e^{-(R+r)Ct}\} \quad (45)$$

$$w_c = \frac{1}{2}CE^2[e^{-(R+r)CT} - 1]e^{-(R+r)Ct} \quad (46)$$

The graphical representation of these equations is given in Fig. 32 for a pulse width that is equal to four times the time constant. These curves show the total amount of energy at any time, measured from  $t = 0$ . The energy supplied by the generator during the pulse interval increases exponentially according to a time constant equal to  $(R + r)C$ . After the time

$t = T$ , the additional energy supplied by the generator is zero, and therefore the total energy supplied up to the time  $t = T$  is the same as the energy from the time  $t = T$  on.

*Energy during Pulse.*—If Eqs. (43) and (44) are added, their sum is seen to be exactly equal to the equation for energy supplied,  $w_v$ . In other words, the sum of the stored energy and dissipated energy is always equal to the energy supplied by the generator.

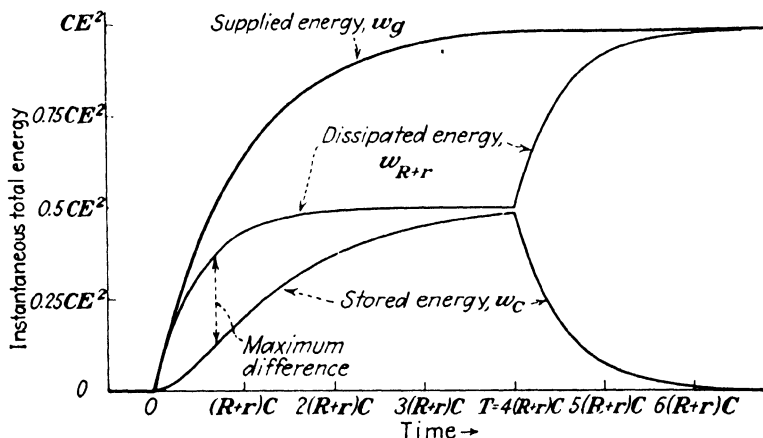


FIG. 32.—Behavior of instantaneous total energy in a basic RC network that is subjected to a rectangular-pulse voltage.

If the pulse width is large compared with the time constant, then at the time  $t = T$  the energy equations become

$$\left. \begin{aligned} w_v &= CE^2 \\ w_{R+r} &= \frac{1}{2}CE^2 \\ w_c &= \frac{1}{2}CE^2 \end{aligned} \right\} t = T \quad \text{and} \quad T \gg (R+r)C$$

because, when  $T \gg (R+r)C$ , both  $e^{-\frac{T}{(R+r)C}}$  and  $e^{-\frac{2T}{(R+r)C}}$  are much less than 1. This is an interesting property of the network; half of the supplied energy is dissipated and half is stored under these special conditions. When the pulse width is not large compared with the time constant, the dissipated and stored energy are not equal even though their sum equals the supplied energy at all times. Figure 32 indicates this while Fig. 33 is a curve showing the variation in the difference between dissipated and stored energy. This difference has a maximum

value of  $\frac{1}{4}CE^2$  at a time equal to  $0.693(R+r)C$ . The values of  $w_c$  and  $w_{R+r}$  at this time are  $\frac{1}{8}CE^2$  and  $\frac{3}{8}CE^2$ , respectively.<sup>1</sup>

*Energy after Pulse.*—In the time interval  $t = T$  to  $t = \infty$ , whatever energy has been stored in the electric field of the capacitor must be dissipated in  $(R+r)$ . This can be deduced

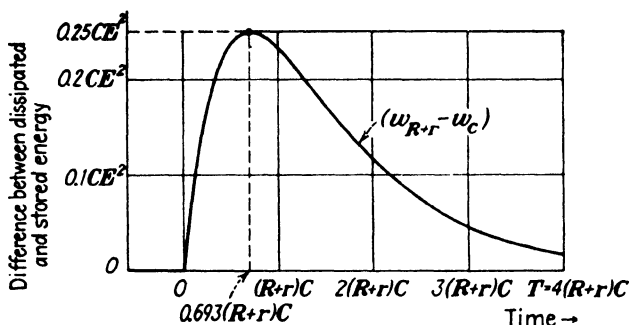


FIG. 33.—Difference between the dissipated and stored energy during the generator pulse.

from energy-conservation principles or from the energy equations, which embody these principles. Rewrite Eq. (45).

$$w_{R+r} = CE^2[1 - e^{-\frac{T}{(R+r)C}}] - \frac{1}{2}CE^2[e^{\frac{T}{(R+r)C}} - 1]^2 e^{-\frac{2t}{(R+r)C}} \quad (45a)$$

The first term is a constant equal to the total energy supplied by the generator for any pulse width  $T$ . The second term, which becomes smaller with time, is exactly equal to the energy stored in the electric field of the capacitance, Eq. (46). Therefore, the energy dissipated in  $(R+r)$  increases by exactly the same amount that the electrostatic energy decreases.

<sup>1</sup> To find the maximum value analytically, differentiate  $(w_{R+r} - w_c)$  with respect to time, set the derivative equal to zero, and insert the resulting value in the original expressions for  $w_{R+r}$  and  $w_c$ .

$$\begin{aligned} \frac{d}{dt}(w_{R+r} - w_c) &= \frac{d}{dt} \left\{ \frac{1}{2} CE^2 [-2e^{-\frac{2t}{(R+r)C}} + 2e^{-\frac{t}{(R+r)C}}] \right\} = 0 \\ e^{-\frac{t}{(R+r)C}} &= \frac{1}{2}; \quad t = 0.693(R+r)C \\ \left. \begin{aligned} w_{R+r} &= \frac{1}{2} CE^2 \left[ 1 - \left(\frac{1}{2}\right)^2 \right] = \frac{3}{8} CE^2 \\ w_c &= \frac{1}{2} CE^2 \left[ 1 - 2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 \right] = \frac{1}{8} CE^2 \end{aligned} \right\} t = 0.693(R+r)C \\ (w_{R+r} - w_c)_{\max} &= \frac{3}{8} CE^2 - \frac{1}{8} CE^2 = \frac{1}{4} CE^2 \end{aligned}$$

To show that all of the energy supplied by the generator is eventually dissipated in  $(R + r)$ , allow  $t$  to approach infinity in Eq. (45a). It is found that

$$w_{R+r} = CE^2[1 - e^{-\frac{T}{(R+r)C}}]$$

when  $t = \infty$ . This is exactly the same amount of energy supplied by the generator during the pulse interval. When  $t = \infty$ , the energy stored in  $C$  is equal to zero.

At the instant the pulse disappears from the generator, the energy in the electric field of the capacitor is a maximum, because from that time on the source of energy is removed. After the generator pulse disappears, a lapse of time equal to twice the time constant results in the dissipation of approximately 98 per cent of this maximum energy in the resistance. This is because the reduction of stored energy (or increase of dissipated energy) is in accordance with a time constant equal to  $\frac{1}{2}(R + r)C$ .

A graphical correlation between the instantaneous-total-energy curves (Fig. 32) and the instantaneous-power curves (Fig. 31) can be made. The integral of the instantaneous-power curve represents the area enclosed under the curve and also represents the instantaneous total energy. The instantaneous total energy supplied by the generator is always equal to the sum of the dissipated and stored energy. Consequently, the sum of the areas under the instantaneous-power curves for the resistance and capacitance is always equal to the area under the instantaneous-power curve for the generator.

*Energy Summary.*—A summary of the energy relations on the basis of Fig. 32 may be worth while. From these curves, the following pertinent features should now be evident:

1. The generator supplies energy only during the pulse interval.
2. The energy supplied by the generator can be as much as  $CE^2$  if the pulse width is large compared with the time constant.
3. The sum of the dissipated and stored energy is always equal to the energy supplied by the generator.
4. For a large pulse width the dissipated energy and stored energy are equal and of value  $\frac{1}{2}CE^2$ .
5. For a short pulse width, more energy is dissipated than is stored.

6. All of the generator energy is eventually dissipated in the resistance.

### BASIC $RC$ NETWORK WITH BOTH RESISTANCE AND CAPACITANCE ACROSS OUTPUT

The two series networks in Figs. 12 and 22 are used extensively in many applications, but variations on series  $RC$  networks are also common. It is interesting to study other series combinations to see how the differential-equation method of analysis applies, to discover what other behavior is possible when a rectangular-pulse voltage is applied, and to arrive at a general equation for *any* series  $RC$  network. One additional network will be treated. It is shown in Fig. 34.

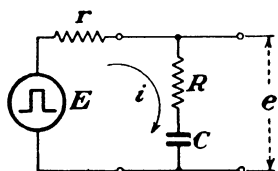


FIG. 34.—Series  $RC$  network with a rectangular-pulse generator.

**18. Equations for Output Pulse.**—In this network the equations for instantaneous current and instantaneous charge are identical with Eqs. (32), (35), (37), and (39), which are given below for convenience.

#### DURING PULSE

$$q_E = CE[1 - e^{-(R+r)C}] \quad (32)$$

$$i_E = \frac{E}{R+r} e^{-(R+r)C} \quad (37)$$

#### AFTER PULSE

$$q_0 = CE[e^{(R+r)C} - 1]e^{-(R+r)C} \quad (35)$$

$$i_0 = -\frac{E}{R+r} [e^{(R+r)C} - 1]e^{-(R+r)C} \quad (39)$$

It is necessary to add the instantaneous voltages developed across  $R$  and  $C$  in order to obtain expressions for the instantaneous output voltage. Since  $e_C = q/C$  and  $e_R = Ri$ , then clearly

#### DURING PULSE

$$e_{C_E} = E[1 - e^{-(R+r)C}]$$

$$e_{R_E} = \frac{ER}{R+r} e^{-(R+r)C}$$

#### AFTER PULSE

$$e_{C_0} = E[e^{(R+r)C} - 1]e^{-(R+r)C}$$

$$e_{R_0} = -\frac{ER}{R+r} [e^{(R+r)C} - 1]e^{-(R+r)C}$$

The sum of  $e_C$  and  $e_R$  is accordingly

$$\begin{array}{c} \text{DURING PULSE} \\ e_E = E \left[ 1 - \frac{r}{R+r} \epsilon^{-\frac{t}{(R+r)C}} \right] \end{array} \quad (47)$$

$$\begin{array}{c} \text{AFTER PULSE} \\ e_0 = \frac{Er}{R+r} \left[ \epsilon^{\frac{T}{(R+r)C}} - 1 \right] \epsilon^{-\frac{t}{(R+r)C}} \end{array} \quad (48)$$

**19. Familiarization with Output-pulse Equations.**—In order to attach physical significance to these equations before proceeding with their general meaning, it is easy to impose some special conditions on the network that make it possible to obtain the answer just by common sense without the use of Eqs. (47) and (48). Then the same conditions will be inserted into the equations for mathematical verification.

*Example 1.*  $R = 0$ ,  $C = \infty$ .—Suppose that  $R$  and  $C$  are replaced by short circuits; i.e.,  $R = 0$  and  $C = \infty$ .

It is obvious that the output voltage both during and after the pulse will always be zero since the output is completely short-circuited. When this is checked in Eq. (47),

$$e_E = E \left( 1 - \frac{r}{r} \epsilon^{-\frac{t}{r\infty}} \right) = E(1 - \epsilon^0) = E(1 - 1) = 0$$

and in Eq. (48),

$$e_0 = \frac{Er}{r} (\epsilon^{\frac{T}{r\infty}} - 1) \epsilon^{-\frac{t}{r\infty}} = E(\epsilon^0 - 1) \epsilon^0 = E(1 - 1) = 0$$

It is seen that  $e = 0$  always when  $R = 0$  and  $C = \infty$ .

*Example 2.*  $C = \infty$ .—Another supposition is that  $C = \infty$ ; i.e., only the capacitor is replaced by a short circuit. From a physical basis, if  $C = \infty$ , the network reduces to Fig. 35. Evidently the output voltage will be an exact reproduction of the input voltage in shape, but its amplitude will be governed by  $r$  and  $R$ , which comprise a voltage divider. In fact, the output voltage will be  $e_E = ER/(R+r)$  during the pulse and zero at the instant the pulse disappears from the generator. To check this reasoning, insert  $C = \infty$  into Eqs. (47) and (48).

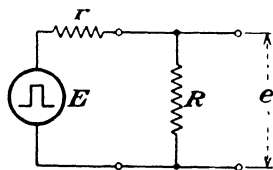


FIG. 35.—Reduction of the network in Fig. 34 to a simple voltage divider when  $C = \infty$ .

$$e_E = E \left[ 1 - \frac{r}{R+r} \epsilon^{-\frac{t}{(R+r)\omega}} \right] = E \left( 1 - \frac{r}{R+r} \right) = \frac{ER}{R+r}$$

$$e_0 = \frac{Er}{R+r} \left[ \epsilon^{\frac{T}{(R+r)\omega}} - 1 \right] \epsilon^{-\frac{t}{(R+r)\omega}} = \frac{Er}{R+r} (\epsilon^0 - 1) \epsilon^{-0} = 0$$

Again the equations contain the special solution.

*Example 3.*  $r = 0$ .—An interesting check is obtained when the generator resistance  $r$  is assumed to be zero. The output voltage must at all times be exactly equal to the generator voltage because it is taken directly across the generator. The

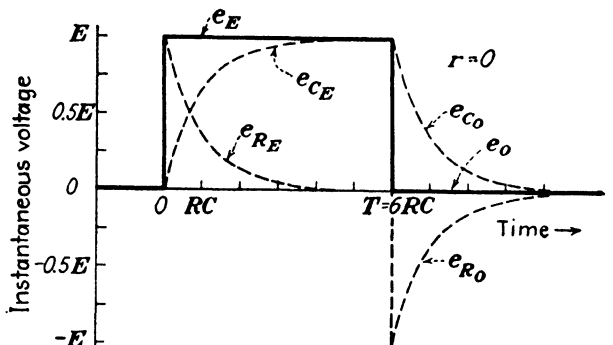


FIG. 36.—Output voltage of the network in Fig. 34 when  $r = 0$ .

conclusion is that during the pulse  $e_E = E$ , and after the pulse  $e_0 = 0$ . To see if the equations contain this information set  $r = 0$  and evaluate  $e$ .

$$e_E = E \left( 1 - \frac{0}{R} \epsilon^{-\frac{t}{RC}} \right) = E(1 - 0) = E$$

$$e_0 = \frac{E \cdot 0}{R} (\epsilon^{\frac{T}{RC}} - 1) \epsilon^{-\frac{t}{RC}} = 0$$

In this check a significant point may be brought out. It has been shown that the sum of  $e_C$  and  $e_R$  is equal to  $E$  during the pulse and equal to zero after the pulse. Nevertheless,  $e_C$  and  $e_R$  considered separately are not of that form. (See Figs. 14 and 24.) The explanation is that during the pulse  $e_{C_1}$  and  $e_{R_1}$  add up to a constant value  $E$ , and after the pulse  $e_{C_0}$  and  $e_{R_0}$  add up to zero at all times. Figure 36 illustrates this. Note that the sum of  $e_C$  and  $e_R$  will be a rectangular voltage for any pulse width.

While these special examples do not show the complete generality of Eqs. (47) and (48), the equations are general, nevertheless, since no special values of  $r$ ,  $R$ , and  $C$  were assumed in their derivation.

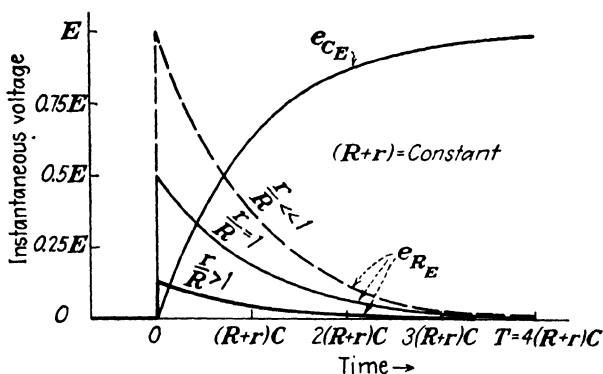


FIG. 37.—Voltage across  $C$  and across  $R$  for the network in Fig. 34 during the generator-pulse interval only. Three curves are shown for different ratios of  $r/R$  but for constant  $(R+r)$ .

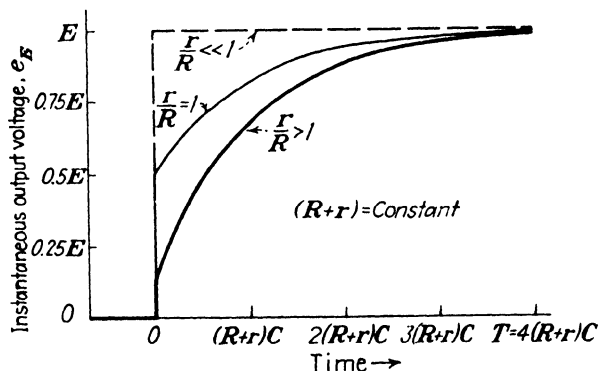


FIG. 38.—Output voltage of the network in Fig. 34 during the generator pulse for three values of  $r/R$  but for constant  $(R+r)$ . These curves result from the addition of  $e_{CE}$  and  $e_{RE}$  in Fig. 37.

**20. Network Behavior.**—It is now appropriate to discuss the network more generally under conditions where  $r$ ,  $R$ , and  $C$  have values different from zero or infinity. Instead of considering Eqs. (47) and (48), it is clearer to consider  $e_C$  and  $e_R$  separately and then to take their sum. First, consider the interval during which the pulse exists at the generator terminals. The voltage across  $R$  and  $C$  in this interval is given in Fig. 37, where

the pulse duration is large compared with  $(R + r)C$ . A family of curves has been drawn for  $e_{R_0}$  with different ratios of  $r/R$  but holding  $(R + r)$  constant. By graphical addition of  $e_{C_0}$

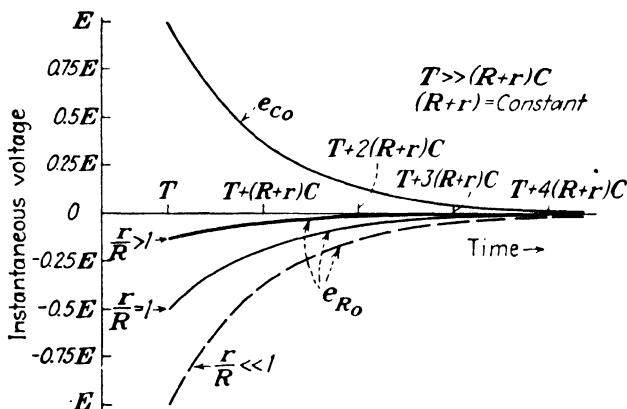


FIG. 39.—Voltage across  $C$  and across  $R$  for the network in Fig. 34 after the generator pulse disappears. The pulse width is large compared with the time constant. Three curves are shown for different ratios of  $r/R$  but for constant  $(R + r)$ .

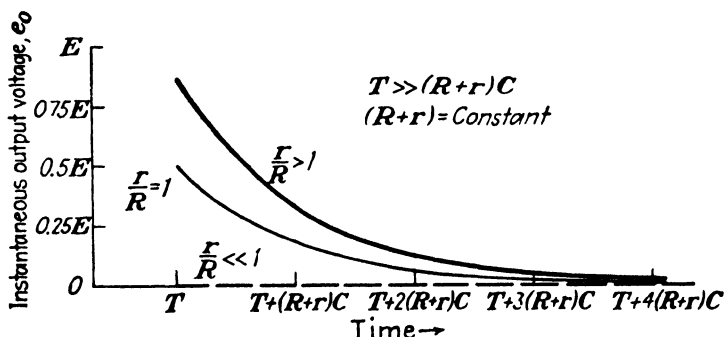


FIG. 40.—Output voltage of the network in Fig. 34 for three values of  $r/R$  but for constant  $(R + r)$ , after the generator pulse disappears. The pulse width is large compared with the time constant. These curves result from the addition of  $e_{C_0}$  and  $e_{R_0}$  in Fig. 39.

and  $e_{R_0}$  the output voltage wave form can be obtained. This has been done in Fig. 38.

After the pulse disappears from the generator, the voltage across  $R$  and  $C$  is as indicated in Fig. 39 where it has been assumed that the pulse duration is large compared with  $(R + r)C$ . Several curves for different ratios of  $r/R$  but fixed  $(R + r)$

have been drawn for  $e_{R_0}$ . The graphical sum of the curves is shown in Fig. 40.

The complete wave form of  $e$  depends upon the ratio  $r/R$  and upon the relative values of  $T$  and  $(R+r)C$ . Various output voltages are indicated in Fig. 41 for different conditions. The

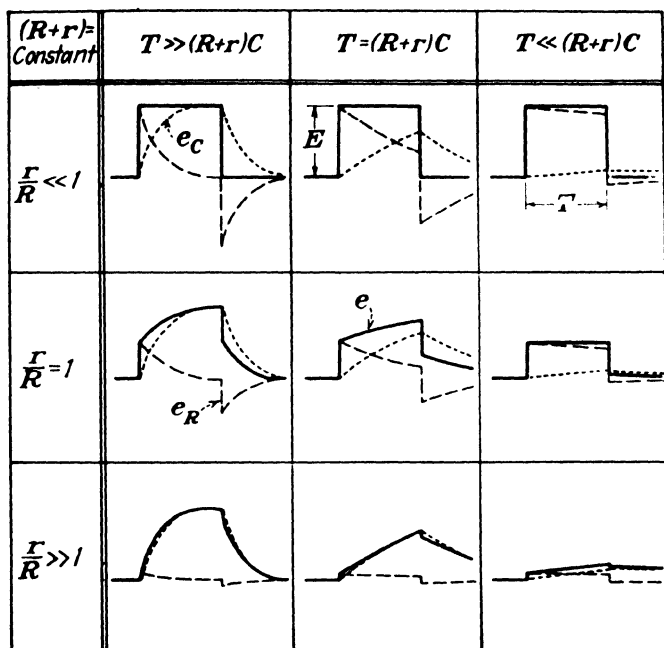


FIG. 41.—Complete output voltage for the network in Fig. 34. The output voltage depends upon  $r/R$  and the relative values of  $T$  and  $(R+r)C$ .

dotted lines represent  $e_c$  and  $e_R$ , and the solid lines indicate the algebraic sum of  $e_c$  and  $e_R$ , which is the output voltage.

### SUMMARY AND GENERALIZATION OF RESULTS

To summarize this chapter and to generalize the results, it is illuminating to reflect upon the three series networks that have been analyzed so far. The three networks are shown in Figs. 12, 22, and 34. Each has been analyzed for output voltage as a function of time for perfectly general values of  $r$ ,  $R$ , and  $C$ . Is it possible to extend the analysis to *any* series network that contains only resistance and capacitance? The considerations that follow demonstrate that it is possible and, moreover, that the analyses already made will pertain equally

well to any series network containing resistance and capacitance only, if the equations are judiciously applied.

**21. Example of Extension of Results.**—As an illustrative example of the extension of the results to other networks, suppose the output voltage as a function of time is required for the network in Fig. 42. By combining all resistors and all capacitors into a single equivalent resistance and a single equivalent capacitance, the network in Fig. 43 results (Chap. I, page 14). This equivalent network will produce exactly the same current flow as that of Fig. 42, and the capacitor  $C_s$  will acquire the same charge as  $C_1$  and  $C_2$ , even though the identity of the output terminals has been lost.

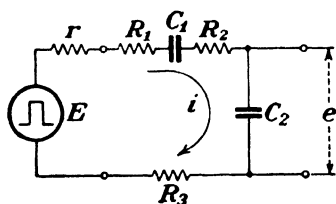


FIG. 42.—Series RC network with a rectangular-pulse generator.

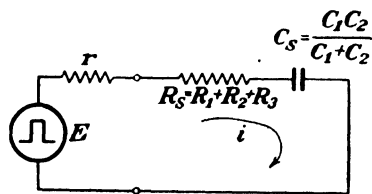


FIG. 43.—Equivalent series network for the network in Fig. 42.

Now the flow of charge for the series network shown in Fig. 43 is given by Eq. (32) during the pulse interval and by Eq. (35) after the pulse disappears, because these equations apply for *any* value of resistance and capacitance. In the case of the equivalent network in Fig. 43, these equations become

$$q_E = C_s E \left[ 1 - e^{-\frac{t}{(R_s + r)C_s}} \right] \quad (32a)$$

$$q_0 = C_s E \left[ e^{\frac{T}{(R_s + r)C_s}} - 1 \right] e^{-\frac{t}{(R_s + r)C_s}} \quad (35a)$$

Once these equations are known, the output voltage is readily obtained by dividing the charge by the capacitance  $C_2$ , which is connected across the output in the actual network. The output voltage, therefore, is

$$e_E = \frac{C_s E}{C_2} \left[ 1 - e^{-\frac{t}{(R_s + r)C_s}} \right]$$

$$e_0 = \frac{C_s E}{C_2} \left[ e^{\frac{T}{(R_s + r)C_s}} - 1 \right] e^{-\frac{t}{(R_s + r)C_s}}$$

where  $R_s = R_1 + R_2 + R_3$  and  $C_s = C_1 C_2 / (C_1 + C_2)$ .

In a similar manner the equations already derived can be used to obtain the output voltage for any series network containing resistance and capacitance only.

**22. General Series RC Network.**—The single general network that represents any conceivable arrangement of series resistors and capacitors is given in Fig. 44. The values of  $R'$  and  $C'$  are unrestricted. Either  $R'$  or  $C'$  or both may be zero, infinity, or even the equivalent resistance and capacitance

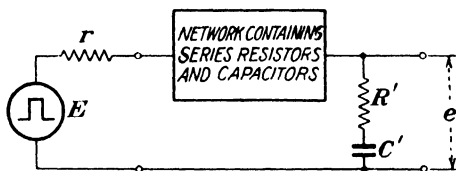


FIG. 44.—A general series  $RC'$  network with a rectangular-pulse generator.

of any number of series resistors and capacitors across the output terminals.

Evidently the equations for the output voltage of this network will also be general. These equations are obtained merely by extension of Eqs. (47) and (48). To make clear the derivation of the general equations, the series equivalent network of Fig. 44 is presented in Fig. 45. The equations for instantaneous current and instantaneous charge in this equivalent network are

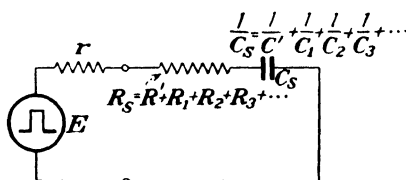


FIG. 45.—Equivalent series network for that in Fig. 44.

Eqs. (32), (35), (37), and (39), which are valid for any values whatsoever of resistance and capacitance. These equations, written for the general values of resistance and capacitance, are

DURING PULSE

$$q_E = C_s E [1 - e^{-\frac{t}{(R_s + r)C_s}}] \quad (32')$$

$$i_E = \frac{E}{R_s + r} e^{-\frac{t}{(R_s + r)C_s}} \quad (37')$$

## AFTER PULSE

$$q_0 = C_s E [\epsilon^{\frac{T}{(R_s+r)C_s}} - 1] \epsilon^{-\frac{t}{(R_s+r)C_s}} \quad (35')$$

$$i_0 = -\frac{E}{R_s + r} [\epsilon^{\frac{T}{(R_s+r)C_s}} - 1] \epsilon^{-\frac{t}{(R_s+r)C_s}} \quad (39')$$

The general output voltage will be the algebraic sum of the voltage across  $C'$  and across  $R'$ . The voltages to be summed are obtained directly from the relations  $e_{C'} = q/C'$  and  $e_{R'} = R'i$ .

## DURING PULSE

$$e_{C's} = \frac{C_s E}{C'} [1 - \epsilon^{-\frac{t}{(R_s+r)C_s}}]$$

$$e_{R's} = \frac{ER'}{R_s + r} \epsilon^{-\frac{t}{(R_s+r)C_s}}$$

## AFTER PULSE

$$e_{C'o} = \frac{C_s E}{C'} [\epsilon^{\frac{T}{(R_s+r)C_s}} - 1] \epsilon^{-\frac{t}{(R_s+r)C_s}}$$

$$e_{R'o} = -\frac{ER'}{R_s + r} [\epsilon^{\frac{T}{(R_s+r)C_s}} - 1] \epsilon^{-\frac{t}{(R_s+r)C_s}}$$

The algebraic sum of  $e_{C'}$  and  $e_{R'}$  yields the general equations for output voltage, which are

$$e_E = \frac{C_s E}{C'} \left\{ 1 + \left[ \frac{R'C'}{C_s(R_s + r)} - 1 \right] \epsilon^{-\frac{t}{(R_s+r)C_s}} \right\} \quad (49)$$

$$e_0 = \frac{C_s E}{C'} [\epsilon^{\frac{T}{(R_s+r)C_s}} - 1] \left[ 1 - \frac{R'C'}{C_s(R_s + r)} \right] \epsilon^{-\frac{t}{(R_s+r)C_s}} \quad (50)$$

These equations represent in mathematical form a complete summary of the output-voltage analysis of this chapter, because they contain the solution for any series network containing resistance and capacitance only.

As an illustration of the use of these general equations, return to the network in Fig. 22 and recall that the equations for the output pulse were found to be

$$e_E = \frac{ER}{R + r} \epsilon^{-\frac{t}{(R+r)C}} \quad (38)$$

$$e_0 = -\frac{ER}{R + r} [\epsilon^{\frac{T}{(R+r)C}} - 1] \epsilon^{-\frac{t}{(R+r)C}} \quad (40)$$

The parameters of the general network in Figs. 44 and 45 become, in the case of the network in Fig. 22,

$$\begin{aligned}C' &= \infty \\R_s &= R' = R \\C_s &= C\end{aligned}$$

These values substituted, the general equations become

$$\begin{aligned}e_k &= C_s E \left\{ \frac{1}{C'} + \left[ \frac{R'}{C_s(R_s + r)} - \frac{1}{C'} \right] \epsilon^{-\frac{t}{(R_s + r)C_s}} \right\} \\&= CE \left\{ \frac{1}{\infty} + \left[ \frac{R}{C(R + r)} - \frac{1}{\infty} \right] \epsilon^{-\frac{t}{(R + r)C}} \right\} \\&= \frac{ER}{R + r} \epsilon^{-\frac{t}{(R + r)C}}\end{aligned}\tag{49a}$$

$$\begin{aligned}e_0 &= C_s E \left[ \epsilon^{\frac{T}{(R_s + r)C_s}} - 1 \right] \left[ \frac{1}{C'} - \frac{R'}{C_s(R_s + r)} \right] \epsilon^{-\frac{t}{(R_s + r)C_s}} \\&= CE \left[ \epsilon^{\frac{T}{(R + r)C}} - 1 \right] \left[ \frac{1}{\infty} - \frac{R}{C(R + r)} \right] \epsilon^{-\frac{t}{(R + r)C}} \\&= -\frac{ER}{R + r} \left[ \epsilon^{\frac{T}{(R + r)C}} - 1 \right] \epsilon^{-\frac{t}{(R + r)C}}\end{aligned}\tag{50a}$$

Equations (49a) and (50a) are the same as Eqs. (38) and (40).

**23. Conclusion.**—In concluding this chapter it is well to examine some of the broader aspects of what has been accomplished. It has been shown for a simple type of network that a general solution by ordinary differential equations is possible. Further, it has been demonstrated in some special cases that the results are applicable to generator voltages of any shape whatsoever. The major pulse-response characteristics of the network have been uncovered by means of this analysis along with an examination of what is actually happening in the network.

A fundamental point concerning the transient network behavior should be emphasized. As was mentioned in Chap. I (page 4), the *nature* of the transient that results from a sudden change is independent of the disturbing force. The magnitude of the transient, however, does depend upon the disturbing force. If any output voltage equation in this chapter is examined, it will be found that the transient is invariably exponential in nature, irrespective of the value of  $E$ , and that this exponential behavior is governed by the network parameters and not by the generator voltage.

It is well to understand this chapter thoroughly because the same underlying principles that have been introduced will be used repeatedly. Furthermore, this chapter, along with Chaps. IV and V, comprises the foundation for the analysis of series-parallel networks.

### Problems

**Prob. 1.** The parameters of the network in Fig. 12 have the following values:  $E = 200$  volts,  $T = 0.015$  sec.,  $r = 4,000$  ohms,  $R = 6,000$  ohms, and  $C = 0.5 \mu\text{f}$ .

- Find the instantaneous charge and instantaneous current at  $t = 0$ .
- What is the time constant?
- Find the instantaneous charge and instantaneous current at  $t = T$ .
- What is the maximum output voltage?
- At what two values of  $t$  is the output voltage equal to 180 volts?

**Prob. 2.** The parameters of the network in Fig. 22 have the following values:  $E = 14$  volts,  $T = 0.03$  sec.,  $r = 4,000$  ohms,  $R = 10,000$  ohms, and  $C = 0.25 \mu\text{f}$ .

- What is the maximum positive output voltage?
- What is the output voltage at  $t = 0.0307$  sec.?
- What is the output voltage 0.03 sec. earlier than the time in b?

**Prob. 3.** The parameters of the network in Fig. 22 are the same as those given in Prob. 2 with the exception of the pulse width  $T$ , which is variable.

- What value of  $T$  is required in order that the positive output voltage at  $t = T$  be exactly equal in magnitude to the negative output voltage at  $t = T$ ?
- What is the magnitude of the output voltage in (a) at  $t = T$ ?

**Prob. 4.** The parameters of the network in Fig. 22 have the following values:  $E = 50$  volts,  $r = 100$  ohms,  $R = 400$  ohms, and  $C = 0.05 \mu\text{f}$ . The output pulse at  $t = T$  has a positive amplitude equal to 36.2 volts.

- What is the generator-pulse width?
- What is the most negative value of output voltage?

**Prob. 5.** Refer to the network in Fig. 12.

- What is the ratio of the steady-state value of charge on  $C$  to  $q_{P\max}$ , the charge on  $C$  at the instant the power delivered to  $C$  is a maximum?
- What is the ratio of the current at  $t = 0$  to  $i_{P\max}$ , the current at the instant the power delivered to  $C$  is a maximum?

**Prob. 6.** In the network in Fig. 12, the time constant is 0.1 sec., the generator-pulse width is 0.01 sec., and  $E = 100$  volts.

- If the voltage increased linearly during the generator pulse in accordance with its initial slope, what would be the value of output voltage at  $t = T$ ?
- What is the actual value of output voltage at  $t = T$ ?

**Prob. 7.** In the network in Fig. 12 it is desired to prevent the output voltage from exceeding 1 per cent of the generator voltage. What is the largest ratio of pulse width to time constant that is tolerable?

**Prob. 8.** A series  $RC$  network contains three resistors of values 300, 400, and 800 ohms, and three capacitors. The generator-rectangular-pulse voltage is 40 volts. The internal resistance of the generator is 500 ohms. What is the maximum value of the voltage that appears across the 400-ohm resistor?

**Prob. 9.** A series  $RC$  network has a time constant equal to 0.1 sec. A rectangular-pulse voltage is applied to the network. The stored energy reaches a maximum value equal to 5 per cent of the total energy supplied by the generator. What is the generator-pulse width?

## CHAPTER IV

### SERIES NETWORKS CONTAINING RESISTANCE AND INDUCTANCE

In the preceding chapter series networks containing resistance and capacitance were analyzed by differential equations for a rectangular-pulse applied voltage. The analysis of three basic series networks led to the generality that any combination of series resistors and capacitors could be analyzed by the same method and by the use of generalized equations. This chapter will develop a similar analysis for series networks containing resistance and inductance. Three basic networks will be analyzed in detail, and then the results will be generalized to include all possible series networks containing resistance and inductance only.

#### BASIC *RL* NETWORK WITH RESISTANCE ACROSS OUTPUT

The first network to be considered is given in Fig. 46, where a single rectangular pulse is applied. The solution for output voltage must be obtained in two steps corresponding to the time intervals  $t = 0$  to  $t = T$ , and  $t = T$  to  $t = \infty$ . The general method is to set up the differential equation for each time interval, and to obtain from each differential equation the equation for output voltage as a function of time.

**1. Equation for Output Pulse ;  $t = 0$  to  $t = T$ .**—To determine the differential equation that applies during the pulse interval, it is necessary to equate the sum of the instantaneous voltages around the network to zero. If  $i_E$  is the instantaneous current that flows, the instantaneous voltages around the network during the pulse interval are

Generator voltage =  $E$

Instantaneous voltage across internal resistance =  $-ri_E$

Instantaneous voltage across inductance =  $-L (di_E/dt)$

Instantaneous voltage across resistance =  $-Ri_E$

The negative sign indicates that the polarity of the voltage is the reverse of the generator-voltage polarity. The generator voltage is the only one that is constant since  $i_E$  varies with time as long as the transient exists. (The terminal voltage of the generator is not constant, however, the terminal voltage being  $E - ri_E$ .) When the algebraic sum of all the individual voltages around the network is equated to zero, the differential equation that applies during the pulse interval is

$$E - ri_E - L \frac{di_E}{dt} - Ri_E = 0 \quad (51)$$

This is a first-order, first-degree equation that can be solved by the method of separation of variables. The equation can be rewritten by grouping terms and separating  $i_E$  and  $t$ , the two variables.

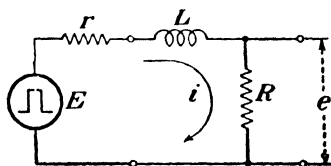


FIG. 46.—Basic series RL network with a rectangular-pulse generator. Resistance output.

$$\frac{dt}{L} = \frac{di_E}{E - (R + r)i_E}$$

Integrate both sides of this equation.

$$\frac{1}{L} \int dt = - \frac{1}{(R + r)} \int \frac{-(R + r) di_E}{E - (R + r)i_E}$$

The solution is

$$-(R + r) \frac{t}{L} = \ln [E - (R + r)i_E] + K_1 \quad (52)$$

To evaluate the constant of integration  $K_1$ , the initial conditions in the network must be utilized. Before the time  $t = 0$ , the current is zero. At the instant  $t = 0$ , the generator pulse has just arrived and has a value  $E$ ; however, the presence of inductance in the network tends to oppose any change in current, so at this instant  $i_E$  is still zero. A mathematical explanation of the fact that  $i_E = 0$  at  $t = 0$  can be made by showing that the rate of change of current is finite. If this is true, then the current cannot change instantly. To demonstrate this, rearrange Eq. (51).

$$\frac{di_E}{dt} = \frac{E - (R + r)i_E}{L}$$

Since  $E$  and  $(R + r)$  are finite, and  $L$  is not zero, then  $di_E/dt$  must be finite. Consequently,  $i_E = 0$  at  $t = 0$ . Setting  $i_E = 0$  and  $t = 0$  in Eq. (52) results in an evaluation of  $K_1$ .

$$K_1 = -\ln E$$

When this value of  $K_1$  is inserted into Eq. (52), the solution becomes

$$-(R + r) \frac{t}{L} = \ln [E - (R + r)i_E] - \ln E$$

This can be written

$$-(R + r) \frac{t}{L} = \ln \left[ \frac{E - (R + r)i_E}{E} \right]$$

Convert to the exponential form.

$$\epsilon^{-\frac{(R+r)t}{L}} = \frac{E - (R + r)i_E}{E}$$

Solve for  $i_E$ .

$$i_E = \frac{E}{R + r} [1 - \epsilon^{-\frac{(R+r)t}{L}}] \quad (53)$$

The voltage at the output terminals is  $Ri_E$ . Therefore, the solution for the output voltage as a function of time during the pulse interval is

$$e_E = Ri_E = \frac{ER}{R + r} [1 - \epsilon^{-\frac{(R+r)t}{L}}] \quad (54)$$

Equation (54) represents a complete mathematical statement of the behavior of output voltage with time during the pulse interval. This equation can be simplified if the internal resistance of the generator is negligible compared with  $R$ . Equation (54) then becomes

$$e_E \approx E(1 - \epsilon^{-\frac{Rt}{L}}) \quad (54a)$$

**2. Network Behavior;  $t = 0$  to  $t = T$ .**—An interpretation of Eqs. (53) and (54) in descriptive and graphical terms is useful to investigate the physical nature of the network behavior. From a discussion of instantaneous current in the network the output-voltage behavior can be deduced.

At the instant the pulse arrives at the generator, current begins to flow. The inductance tends to prevent a change in

the flow of current, and the growth of current in the network increases according to Eq. (53), which is illustrated in Fig. 47. Figure 47 indicates that the current increases exponentially and approaches a value  $E/(R + r)$ . The time constant of the network, which is the time required for the current to change from

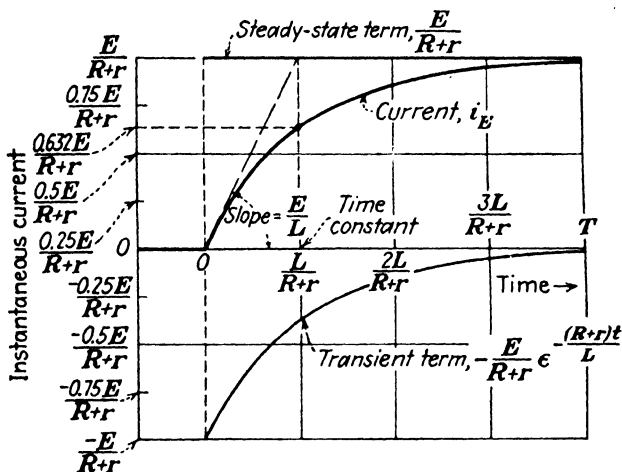


FIG. 47.—Exponential growth of current for a pulse width large compared with  $L/(R + r)$ . The sum of the transient and steady-state terms at any instant equals  $i_E$ .

zero to 63.2 per cent of  $E/(R + r)$ , is equal to  $L/(R + r)$ . At  $t = L/(R + r)$  the instantaneous current, from Eq. (53), is

$$(i_E)_{t=\frac{L}{R+r}} = \frac{E}{R+r} (1 - e^{-1}) = 0.632 \frac{E}{R+r}$$

The dimension of inductance divided by resistance is time (Chap. I, page 19).

The initial rate of change of current can be obtained from Eq. (51) by setting  $i_E$  and  $t$  equal to zero, since the current is zero at  $t = 0$ .

$$\left( \frac{di_E}{dt} \right)_{t=0} = \frac{E}{L}$$

If this rate of change of current were maintained, the steady-state value  $E/(R + r)$  would be reached in a time equal to  $L/(R + r)$ , the time constant. This is indicated graphically in Fig. 47. However, the initial rate of change of current is not

maintained and the instantaneous current reaches the value  $E/(R+r)$  only after infinite time. When the pulse duration is four times the time constant, the current will be slightly more than 98 per cent of  $E/(R+r)$ .

An examination of Eq. (53) reveals the existence of two distinct terms:  $\frac{E}{(R+r)}$ , which is constant, and  $-\frac{E}{R+r}e^{-\frac{(R+r)t}{L}}$ , which decreases in magnitude as time increases. These are the steady-

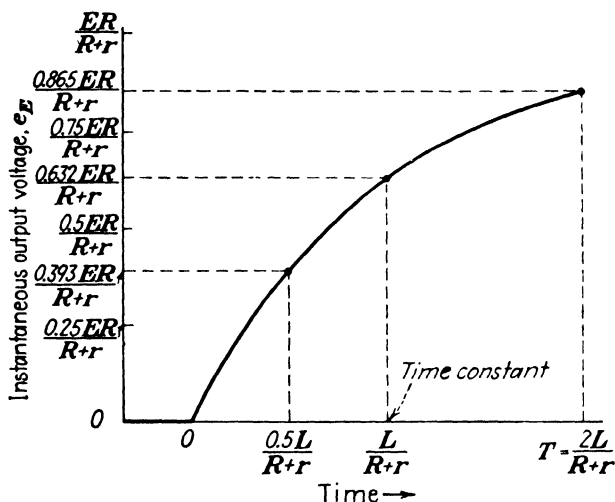


FIG. 48.—Output voltage of the network in Fig. 46 during the generator-pulse interval only.

state and transient terms that are indicated in Fig. 47. Their algebraic sum equals the instantaneous current.

The output voltage will vary in the same manner as the current, since the current through  $R$  and the voltage produced across  $R$  are directly proportional. Figure 48, which is derived from Eq. (54), represents the output voltage as a function of time during the pulse interval when the pulse interval is twice the time constant.

Equation (54) can be used to compute the value of the output voltage at the instant the pulse disappears from the generator. For instance, if the pulse width is one-half the time constant, the output voltage at the instant the pulse disappears from the generator will be

$$(e_E)_{t=\frac{1}{2}\frac{L}{R+r}} = \frac{ER}{R+r} (1 - e^{-\frac{1}{2}}) = 0.393 \frac{ER}{R+r}$$

**3. Equation for Output Pulse;  $t = T$  to  $t = \infty$ .**—To obtain the equation for output voltage for the time interval  $t = T$  to  $t = \infty$ , which is the time interval starting at the instant the generator pulse disappears, the differential equation for this interval must be found. The necessary equation is obtained by setting  $E$  equal to zero in Eq. (51).

$$ri_0 + L \frac{di_0}{dt} + Ri_0 = 0$$

This equation can be solved in the same manner as Eq. (51). Separate variables.

$$-(R+r) \frac{dt}{L} = \frac{di_0}{i_0}$$

Integrate to obtain the solution.

$$-(R+r) \frac{t}{L} = \ln i_0 + K_2 \quad (55)$$

The constant  $K_2$  can be evaluated if the value of  $i_0$  can be determined at the time  $t = T$ . From Eq. (53) the required value of  $i_0$  is obtained by equating  $t = T$ .

$$i_{ET} = \frac{E}{R+r} [1 - e^{-\frac{(R+r)T}{L}}]$$

$i_{ET}$  denotes the instantaneous current at the time  $T$ . Therefore,  $K_2$  can be evaluated.

$$K_2 = -(R+r) \frac{T}{L} - \ln \left\{ \frac{E}{R+r} [1 - e^{-\frac{(R+r)T}{L}}] \right\}$$

Substitute this value of  $K_2$  into Eq. (55).

$$-(R+r) \frac{t}{L} = \ln i_0 - (R+r) \frac{T}{L} - \ln \left\{ \frac{E}{R+r} [1 - e^{-\frac{(R+r)T}{L}}] \right\}$$

Collect like terms.

$$-(R+r) \frac{(t-T)}{L} = \ln \left\{ \frac{(R+r)i_0}{E[1 - e^{-\frac{(R+r)T}{L}}]} \right\}$$

After conversion to the exponential form, the equation for  $i_0$  as a function of time becomes

$$i_0 = \frac{E}{R+r} \left[ 1 - e^{-\frac{(R+r)T}{L}} \right] e^{-\frac{(R+r)(t-T)}{L}}$$

which simplifies to

$$i_0 = \frac{E}{R+r} \left[ e^{\frac{(R+r)T}{L}} - 1 \right] e^{-\frac{(R+r)t}{L}} \quad (56)$$

The output voltage for the time interval  $t = T$  to  $t = \infty$  is

$$e_0 = Ri_0 = \frac{ER}{R+r} \left[ e^{\frac{(R+r)T}{L}} - 1 \right] e^{-\frac{(R+r)t}{L}} \quad (57)$$

If the generator resistance is very small compared with  $R$ , then Eq. (57) becomes

$$e_0 \approx E \left( e^{\frac{RT}{L}} - 1 \right) e^{-\frac{Rt}{L}} \quad (57a)$$

If, in addition, the pulse width is very large compared with  $L/R$ , then Eq. (57a) can be further reduced to

$$e_0 \approx E e^{-\frac{R(t-T)}{L}} \quad (57b)$$

Mathematically, the condition that

$$\frac{RT}{L} - 1 \approx \frac{RT}{L}$$

must be true for Eq. (57b) to be valid. This condition will be true when  $\frac{RT}{L} \gg 1$ ; i.e.,  $(RT/L) \gg 1$ , or  $T \gg L/R$ .

**4. Network Behavior;  $t = T$  to  $t = \infty$ .**—At the instant the pulse disappears from the generator, current is no longer supplied to the network. However, the current will not become zero immediately because the inductance tends to maintain the current flow. The current will decrease in accordance with Eq. (56). Suppose, for simplicity, the pulse width is very large compared with  $L/(R+r)$ . This means that at  $t = T$ , the instant the generator pulse disappears, the current in the network will be practically  $E/(R+r)$ , the steady-state value. The current will subsequently decay exponentially as shown in Fig. 49. After a decay time equal to  $L/(R+r)$ , the current will have an

instantaneous value that is 36.8 per cent of its value at  $t = T$ . To verify this recall that the value of  $i_E$  at  $t = T$  is

$$i_{ET} = \frac{E}{R+r} \left[ 1 - e^{-\frac{(R+r)T}{L}} \right]$$

In Eq. (56), when  $t = T + \frac{L}{R+r}$ ,

$$\begin{aligned} (i_0)_{t=T+\frac{L}{R+r}} &= \frac{E}{R+r} \left[ e^{\frac{(R+r)T}{L}} - 1 \right] e^{-\frac{(R+r)T}{L}} e^{-1} \\ &= \frac{E}{R+r} \left[ 1 - e^{-\frac{(R+r)T}{L}} \right] e^{-1} = i_{ET} e^{-1} = 0.368 i_{ET} \end{aligned}$$

Therefore, the instantaneous current decreases to 36.8 per cent of the value it had at  $t = T$  during a decay time equal to the time

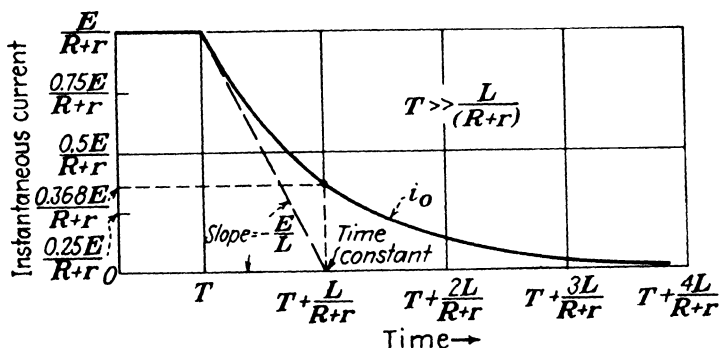


FIG. 49.—Exponential decay of current for a pulse width large compared with  $L/(R+r)$ . The value of current at  $t = T$  equals the steady-state value of current shown in Fig. 47.

constant. This is generally true, and not dependent upon the actual value of  $i_E$  at the time  $t = T$ . Precisely speaking, an infinite length of time must elapse before the current decays to zero. But after a decay time equal to four times the time constant, the current is less than 2 per cent of its initial value at  $t = T$ .

The transient term during the decay of current is completely responsible for all current flow, because the voltage applied to the network from the time  $T$  on is zero, and hence the steady-state term in Eq. (56) is zero.

**5. General Network Behavior.**—Figure 50 shows the output pulse obtained for the network in Fig. 46 when the pulse width

is approximately equal to the network time constant. Figure 51

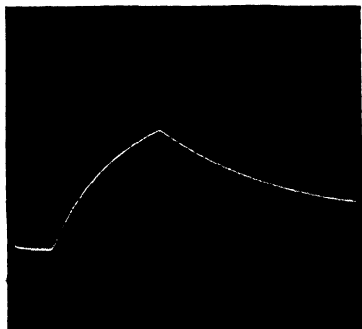


FIG. 50.—Oscillogram of the output pulse for the network in Fig. 46 when the pulse width is comparable to the network time constant.

reveals the complete output voltage versus time for generator pulses of various time durations and equal amplitude. As the network time constant is made smaller compared with the generator-pulse width, the output voltage is seen to approach more and more closely an exact reproduction of the input voltage. This can be qualitatively explained on the basis of the expression for the time constant.

The time constant is directly proportional to the inductance and inversely proportional to the resistance. There are essentially two means of reducing

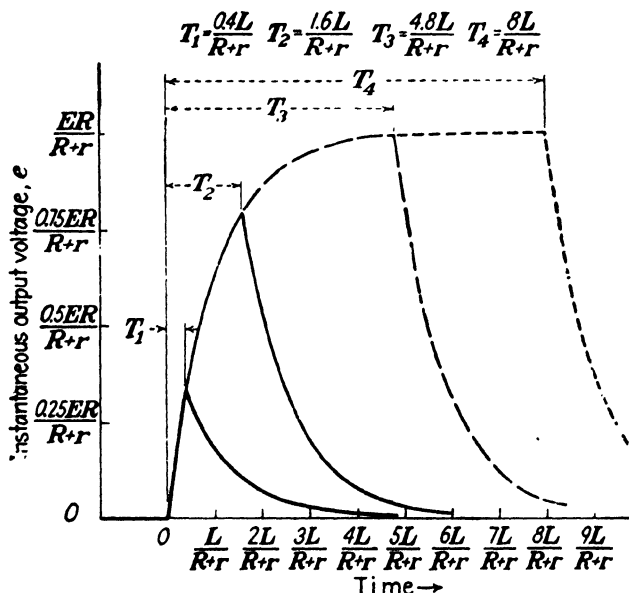


FIG. 51.—Complete output voltage of the network in Fig. 46 for four generator pulses of equal amplitude  $E$  but different pulse widths.

the time constant: reduction of inductance or increase of resistance. If the inductance is reduced, there is less opposition to

establishment of current in the network. On the other hand, if the resistance is increased, a smaller change in current will be required to produce the output voltage pulse and the opposition to this smaller change will be less. In either case, the output voltage, which is proportional to the current, will follow the generator pulse more closely. Figure 52 gives a graphical idea of the relative values of pulse width and time constant necessary to approach various degrees of pulse reproduction.

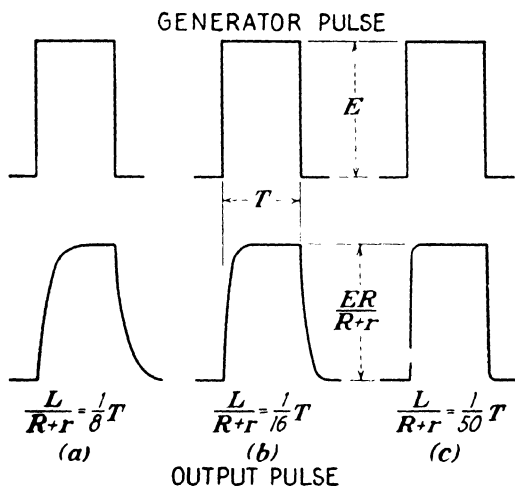


FIG. 52.—The output pulse approaches the shape of the generator pulse as the network time constant is made small compared with the pulse width.

A mathematical examination of the differential equation under the condition that  $L/(R + r)$  is very small compared with the generator-pulse width bears out this qualitative conclusion. For any generator voltage  $e_v$ , Eq. (51) becomes

$$e_v = (R + r)i + L \frac{di}{dt}$$

which can be written

$$\frac{e_v}{R + r} = i + \frac{L}{(R + r)} \frac{di}{dt}$$

The term  $\frac{L}{(R + r)} \frac{di}{dt}$  is negligible compared with  $i$  because (1) the steady-state value of current is attained very quickly when  $L/(R + r)$  is small compared with the generator-pulse width,

and (2)  $di/dt$  becomes small as the steady-state value of current is approached. Therefore,

$$\frac{e_o}{R+r} \approx i$$

Multiply each side of this approximate equation by  $R$ , and remember that  $e = Ri$ .

$$\frac{e_o R}{R+r} \approx Ri = e; \quad \frac{L}{R+r} \ll T$$

This proves that the output voltage  $e$  is approximately proportional to  $e_o$  irrespective of the shape of the generator voltage.

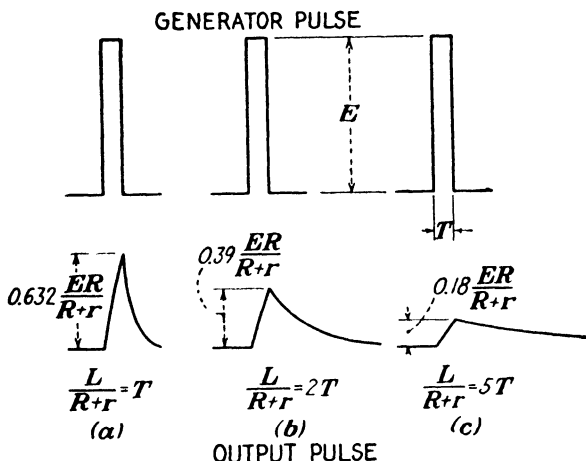


FIG. 53.—The output pulse departs more and more from the generator pulse in both shape and amplitude as the network time constant is made larger compared with the pulse width. If  $L/(R+r)$  is *very* large compared with  $T$ , the output voltage is approximately the integral of the generator pulse.

Figure 51 also indicates that the output voltage becomes less similar to the generator-pulse voltage as the time constant becomes large compared with the pulse width. This behavior can be understood qualitatively by visualizing the situation where the increase of current in the network is so slow that the generator pulse disappears before appreciable current flows. In Fig. 53 the output-voltage departure from the generator pulse in both shape and amplitude is illustrated for three representative cases.

**6. Pulse Integration.**—The ability of this network to produce an output voltage that is approximately the integral of the

generator pulse is indicated by Fig. 53c. To show by graphical means that Fig. 53c is approximately the integral of the generator pulse, the construction of the derivative of the output pulse can be carried out (see Fig. 20, page 52).

Another more general means of showing the property of integration is by considering that any voltage  $e_g$ , a function of time, is applied to this network. From Eq. (51) the equation that sums all of the instantaneous voltages around the network is

$$e_g = (R + r)i + L \frac{di}{dt}$$

which can be written

$$\frac{e_g}{R + r} = i + \frac{L}{(R + r)} \frac{di}{dt}$$

When  $L/(R + r)$  is very large compared with the generator-pulse width,  $i$  is negligible compared with  $\frac{L}{(R + r)} \frac{di}{dt}$ . Therefore,

$$e_g \approx L \frac{di}{dt}$$

Separating variables and integrating will lead to an approximate equation for  $i$ .

$$i \approx \frac{1}{L} \int e_g dt$$

The output voltage is  $Ri$ , and therefore

$$e = Ri \approx \frac{R}{L} \int e_g dt; \quad \frac{L}{R + r} \gg T$$

This proves that the output voltage  $e$  is approximately proportional to the integral of the generator voltage, for a generator voltage of any shape whatsoever.

#### BASIC RL NETWORK WITH INDUCTANCE ACROSS OUTPUT

If the output voltage is taken across the inductance instead of across the resistance in the network of Fig. 46, the network will exhibit different pulse-response characteristics. Figure 54 shows this arrangement.

**7. Equation for Output Pulse;  $t = 0$  to  $t = T$ .**—It is extremely simple to obtain the output-pulse equations for this network because the instantaneous current in the network has already been determined. It is only necessary to recall that the voltage

across the inductance is  $L(di/dt)$ . The time derivative of Eq. (53), which applies during the generator pulse, is

$$\frac{di_E}{dt} = \frac{E}{L} \epsilon^{-\frac{(R+r)t}{L}} \quad (58)$$

So the output voltage for this time interval is

$$e_E = L \frac{di_E}{dt} = E \epsilon^{-\frac{(R+r)t}{L}} \quad (59)$$

If the internal resistance of the generator is negligible compared with the resistance  $R$ , then Eq. (59) becomes

$$e_E \approx E \epsilon^{-\frac{Rt}{L}} \quad (59a)$$

**8. Network Behavior;  $t = 0$  to  $t = T$ .**—In order to attach physical significance to Eqs. (58) and (59), they will be examined in terms of the network in Fig. 54.

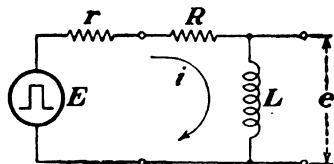


FIG. 54.—Basic series  $RL$  network with a rectangular-pulse generator. Inductance output.

Equation (59) shows that the output voltage is exactly equal to the generator voltage  $E$  at  $t = 0$ . If this is true, then the current at  $t = 0$  must be zero because any flow of current would result in a voltage drop across  $(R + r)$  and would therefore make the output voltage less than  $E$ . The fact that the output voltage equals  $E$  at  $t = 0$  could have been deduced by considering that the inductance opposes any change in current, and that the current in the network cannot change instantly if inductance is present. If the current is zero at  $t = 0$ , all of the generator voltage must appear across the inductance in order to have the instantaneous sum of all voltages around the network be zero.

Although the instantaneous current is zero at  $t = 0$ , the rate of change of current has a value equal to  $E/L$ . This is indicated in both Fig. 47 and Eq. (58). Consequently, at some time after  $t = 0$ , current will be flowing and producing a voltage drop across  $(R + r)$ , thus detracting from the voltage across the output. Another viewpoint also leads to the conclusion that the output voltage decreases with time. Figure 47 reveals that the rate of change of current (the slope of the current curve)

becomes smaller with time. The output voltage that appears across the inductance is directly proportional to this rate of change of current, and consequently the output voltage decreases with time. Figure 55 gives the output voltage during the time interval  $t = 0$  to  $t = T$  where the pulse width is large compared with the time constant. When a time equal to the time constant has elapsed, the output voltage is 36.8 per cent of the value it had at  $t = 0$ . When a time equal to four times the time constant has elapsed, the output voltage is less than 2 per cent of  $E$ .

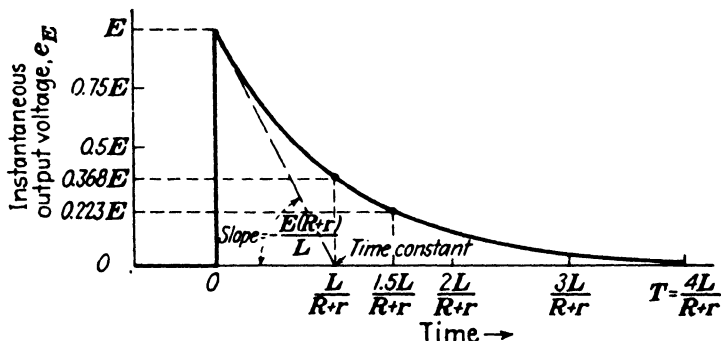


FIG. 55. —Output voltage of the network in Fig. 54 during the generator pulse.

To compute the positive output voltage at the instant the generator pulse disappears, substitution of the pulse width in Eq. (59) is necessary. For instance, suppose the pulse width is equal to 1.5 times the time constant. Then from Eq. (59)

$$(e_E)_{t = \frac{1.5L}{R+r}} = Ee^{-1.5} = 0.223E$$

The steady-state value of output voltage is zero since there is no constant term in Eq. (59); thus Fig. 55 represents the transient term as well as the output voltage.

**9. Equation for Output Pulse;  $t = T$  to  $t = \infty$ .**—The output voltage during the time interval  $t = T$  to  $t = \infty$  can be determined by differentiating Eq. (56) and multiplying by  $L$ . From Eq. (56)

$$\frac{di_0}{dt} = -\frac{E}{L} \left[ \epsilon^{\frac{(R+r)T}{L}} - 1 \right] \epsilon^{-\frac{(R+r)t}{L}} \quad (60)$$

and 
$$e_0 = L \frac{di_0}{dt} = -E \left[ \epsilon^{\frac{(R+r)T}{L}} - 1 \right] \epsilon^{-\frac{(R+r)t}{L}} \quad (61)$$

When  $r \ll R$ , Eq. (61) becomes

$$e_0 \approx -E(\epsilon^{\frac{RT}{L}} - 1)\epsilon^{-\frac{Rt}{L}} \quad (61a)$$

If, in addition, the pulse width is very large compared with the time constant, Eq. (61a) simplifies further.

$$e_0 \approx -E\epsilon^{-\frac{R(t-T)}{L}} \quad (61b)$$

Equation (61b) is valid only when  $\epsilon^{\frac{RT}{L}} - 1 \approx \epsilon^{\frac{RT}{L}}$  and  $r \ll R$ .

**10. Network Behavior;  $t = T$  to  $t = \infty$ .**—To complete the graphical picture of output voltage and to investigate the physical behavior of the network after the pulse disappears from the generator, Eq. (61) will be interpreted.

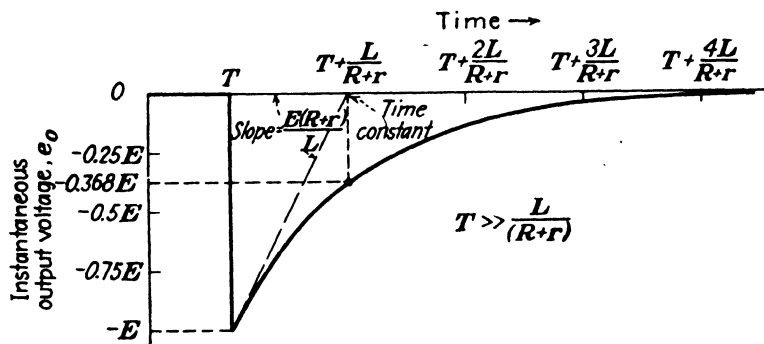


FIG. 56.—Output voltage of the network in Fig. 54 after the generator pulse disappears and for a pulse width large compared with the time constant.

At the instant the generator pulse disappears, the current supplied by the generator becomes zero. However, current continues to flow in the network because the presence of inductance will not permit the current to change instantly. The behavior of current is illustrated in Fig. 49. The rate of change of current, or slope of the instantaneous current curve, is negative and decreases with time. Therefore, the output voltage, which is the inductance times the rate of change of current, will be negative and will also decrease with time. When a voltage changes from positive to negative, it merely indicates reversal of polarity.

At the instant  $t = T$ , the output voltage will have a maximum negative value. One explanation for this can be made by

recognizing that the negative rate of change of current is a maximum at  $t = T$ . A slightly more involved but none the less applicable explanation can be made on the basis of transient and steady-state terms. At  $t = T$  the current in the network is a maximum and therefore departs more from its steady-state value, zero, than at any other time in the interval  $t = T$  to  $t = \infty$ . The transient term always makes up the difference between the existing value of current and the steady-state value.

Figure 56 shows the output voltage as a function of time from the time  $T$  on for a pulse width that is large compared with  $L/(R + r)$ . This is the characteristic exponential decrease of voltage, which has a value equal to 36.8 per cent of  $E$  after a decay time equal to  $L/(R + r)$ , and a value less than 2 per cent of  $E$  after a decay time equal to  $4L/(R + r)$ .

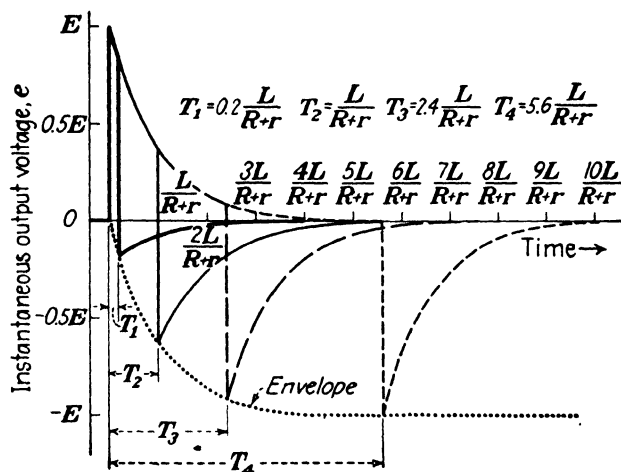


FIG. 57.—Complete output voltage of the network in Fig. 54 for four generator pulses of equal amplitude  $E$  but different pulse widths.

**11. Network Behavior;  $t = 0$  to  $t = \infty$ .**—Equations (59) and (61) can be combined graphically to represent the complete output-pulse shape and amplitude. Figure 57 represents Eqs. (59) and (61) for four generator pulses of equal amplitude  $E$  but with durations equal to 0.2, 1.0, 2.4, and 5.6 times the time constant. These pulse widths have been chosen to illustrate the trends in the output-pulse shape and amplitude for various pulse widths. Before discussing these trends, it is interesting to

observe that the most negative values of output voltage for various pulse widths lie along a definite curve or envelope. The equation of this envelope can be obtained by setting  $t = T$  in Eq. (61) with the following result:

$$e_0 = -E[1 - e^{-\frac{(R+r)T}{L}}]$$

This is the equation of the envelope that is drawn in Fig. 57.

Two illustrations of output pulses obtainable in practice are given in Figs. 58 and 59.



FIG. 58.—Oscilloscope of the output pulse for the network in Fig. 54 when the pulse width is less than the time constant.

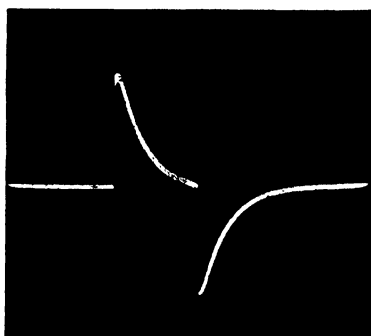


FIG. 59.—Output pulse of the network in Fig. 54 when the pulse width is approximately four times the time constant.

**12. General Network Behavior.**—A study of Fig. 57 reveals some general information about the pulse-response characteristic of the network when the generator-pulse width is changed relative to the network time constant. Two definite trends can be seen: (1) For a generator-pulse width that is short compared with the time constant, the output pulse tends to be a reproduction of the generator pulse, and (2) for a generator pulse that is long compared with the time constant, the output pulse tends to depart more and more from the shape of the generator pulse.

Both of these trends can be explained in mathematical and physical terms. The mathematical explanation for (1) can be made on the basis of Eq. (51), which, if rearranged, becomes

$$\frac{e_g}{R + r} = i + \frac{L}{(R + r)} \frac{di}{dt}$$

when the voltage  $E$  is replaced by a general generator voltage  $e_g$ . If the time constant is large compared with the generator-pulse width, this equation becomes approximately

$$e_g \approx L \frac{di}{dt} = e; \quad T \ll \frac{L}{R+r}$$

which shows clearly that the output voltage is approximately equal to the generator voltage.

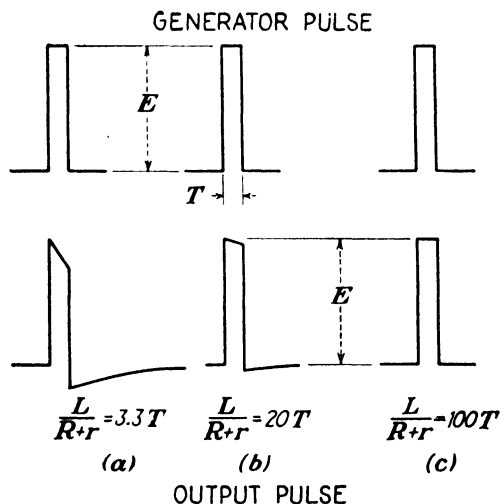


FIG. 60.—The output pulse duplicates the generator pulse more closely in shape and amplitude as the network time constant is made larger compared with the pulse width.

A physical explanation is possible when the effect of a large time constant on the voltage across  $(R + r)$  is considered. The appearance of voltage across  $(R + r)$  is the factor that introduces distortion of the output pulse relative to the generator pulse since the voltage across  $(R + r)$  is the difference between  $E$  and  $e$ . If it can be deduced that the voltage across  $(R + r)$  becomes smaller as the time constant is made larger relative to the pulse width, the necessary explanation will have been found. If the time constant is increased by increasing the inductance, there will be more opposition to change in current, and consequently less current will flow during the pulse interval. Less current flow means less voltage across  $(R + r)$  and hence less output-pulse distortion. If the time

constant is increased by reducing the resistance, less voltage will appear across the resistance for a given current. Again the distortion due to voltage developed across  $(R + r)$  is reduced. Figure 60 presents output pulses for various generator-pulse widths that are small compared with the time constant.

The mathematical explanation for trend (2) is given in the next section. A physical explanation can be made by again applying the principle that voltage produced across  $(R + r)$  is a measure of the output-pulse distortion. If the time constant is made smaller compared with the generator-pulse width, which corresponds to the case where the generator-pulse width is made larger compared with the time constant, the output pulse bears less resemblance to the generator pulse. Suppose the time constant is reduced by reducing  $L$ . Less inductance leads to less opposition to change of current and hence to more current flow during the pulse interval. This results in an increased voltage drop across  $(R + r)$ . If the time constant is reduced by increasing  $(R + r)$ , again there will be an increase in the voltage that appears across  $(R + r)$  during the generator pulse. Either method of reducing the time constant leads to an increase in the voltage across  $(R + r)$ , which indicates that the shape of output pulse is made less similar to the generator-pulse shape. Figure 61 shows output pulses for various generator-pulse widths that are large compared with the time constant.

**13. Pulse Differentiation.**—Figure 61c suggests that the output voltage tends to approach the derivative of the generator voltage as the network time constant is made very small compared with the generator-pulse width. The derivative of the generator-pulse voltage consists of two lines, one at  $t = 0$  extending in the positive direction and one at  $t = T$  extending in the negative direction. Although this network cannot produce an output pulse that is exactly equal to the derivative of the generator pulse, a close approximation is nevertheless possible.

The property of differentiation can be deduced from mathematical as well as graphical considerations and, moreover, for a generator pulse of any shape whatsoever. Rewrite Eq. (51) in terms of a general generator voltage  $e_g$  instead of the rectangular voltage  $E$ .

$$\frac{e_g}{R + r} = i + \frac{L}{(R + r)} \frac{di}{dt}$$

When  $L/(R+r)$  is small compared with the generator-pulse width, the term  $\frac{L}{(R+r)} \frac{di}{dt}$  is negligible compared with  $i$ , and  $i \approx \frac{e_g}{R+r}$ . The derivative of the current times  $L$  is the output voltage. Therefore,

$$L \frac{di}{dt} = e \approx \frac{L}{(R+r)} \frac{de_g}{dt}; \quad \frac{L}{R+r} \ll T$$

This shows that the output voltage is approximately proportional to the derivative of the generator voltage.

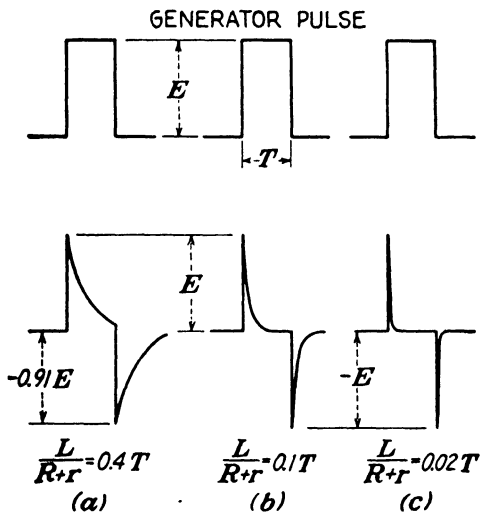


FIG. 61.—The output-pulse shape departs more and more from the generator-pulse shape as the network time constant is made smaller compared with the pulse width. If the network time constant is made sufficiently small, the output voltage is approximately the derivative of the generator pulse.

### POWER AND ENERGY RELATIONS

The power and energy relations for the networks in Figs. 46 and 54 are helpful in obtaining a further understanding of the network behavior. The same power and energy concepts introduced in Chap. III will be applied here.

**14. Energy Transfer and Transformation.**—The generator delivers current to the network during the time interval  $t = 0$  to  $t = T$ . This current flow represents a transfer of energy from

TABLE III.—POWER AND ENERGY RELATIONS IN A BASIC  $RL$  NETWORK

	Time interval $t = 0$ to $t = T$	Time interval $t = T$ to $t = \infty$
Generator voltage, $e_g$	$e_g = E$	$e_g = 0$
Voltage across $R + r$ , $e_{R+r}$	$e_{R+r} = (R + r)i_E$	$e_{R+r} = (R + r)i_0$
Voltage across $L$ , $e_L$	$e_L = L \frac{di_E}{dt}$	$e_L = L \frac{di_0}{dt}$
Generator power, $p_g$	$p_g = E i_E$	$p_g = 0$
Resistance power, $p_{R+r}$	$p_{R+r} = (R + r)i_E^2$	$p_{R+r} = (R + r)i_0^2$
Inductance power, $p_L$	$p_L = i_E L \frac{di_E}{dt}$	$p_L = i_0 L \frac{di_0}{dt}$
Supplied energy (from generator), $w_g$	$w_g = \int_0^T E i_E dt$	$w_g = 0$
Dissipated energy (in resistance), $w_{R+r}$	$w_{R+r} = \int_0^T (R + r) i_E^2 dt$	$w_{R+r} = \int_T^\infty (R + r) i_0^2 dt$
Stored energy (in inductance), $w_L$	$w_L = \int_0^{iT} L i_E di_E$	$w_L = \int_{iT}^\infty L i_0 di_0$
Instantaneous current, $i$	$i_E = \frac{E}{R + r} \left[ 1 - e^{-\frac{(R+r)t}{L}} \right]$	$i_0 = \frac{E}{R + r} \left[ e^{-\frac{(R+r)T}{L}} - 1 \right] e^{-\frac{(R+r)t}{L}}$
Instantaneous rate of change of current, $di/dt$	$\frac{di_E}{dt} = \frac{E}{L} e^{-\frac{(R+r)t}{L}}$	$\frac{di_0}{dt} = -\frac{E}{L} \left[ e^{-\frac{(R+r)T}{L}} - 1 \right] e^{-\frac{(R+r)t}{L}}$

the generator to the network. Some of the energy is stored in the magnetic field of the inductance, and some is dissipated in the resistance. After the generator pulse disappears, no additional energy is supplied to the network. The stored energy in the inductance is gradually released and dissipated in the resistance. This release of energy is accomplished through the medium of current flow. Eventually, the stored energy becomes negligible, and all of the energy that the generator supplied has been dissipated in the resistance. This is a qualitative picture of the energy relations. Mathematical analysis shows up many details which are otherwise obscured.

#### 15. Instantaneous Power and Instantaneous Total Energy.—

Applying the basic power and energy relations, Eqs. (41) and (42), to this network will yield an exact description of the network behavior. Table III is a tabulation of the relations that will be required for this analysis. The equations in this table have either been previously derived or can be obtained by direct substitution into the fundamental power and energy equations.

**16. Power Relations.**—First, the power relations will be considered. Direct substitution of the equations for  $i$  and  $di/dt$  into the power equations yields

#### DURING PULSE

$$p_g = \frac{E^2}{R+r} \left[ 1 - e^{-\frac{(R+r)t}{L}} \right] \quad (62)$$

$$p_{R+r} = \frac{E^2}{R+r} \left[ 1 - 2e^{-\frac{(R+r)t}{L}} + e^{-\frac{2(R+r)t}{L}} \right] \quad (63)$$

$$p_L = \frac{E^2}{R+r} \left[ e^{-\frac{(R+r)t}{L}} - e^{-\frac{2(R+r)t}{L}} \right] \quad (64)$$

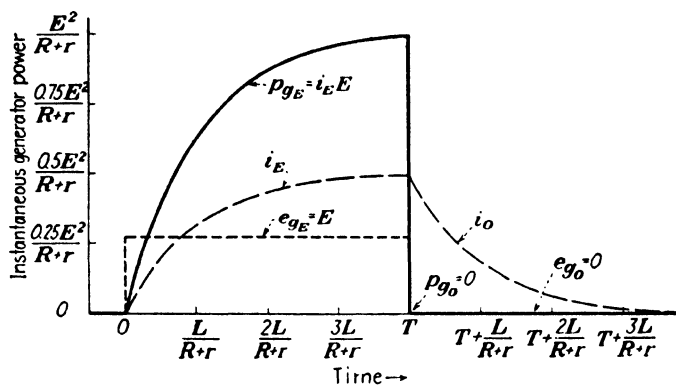
#### AFTER PULSE

$$p_g = 0$$

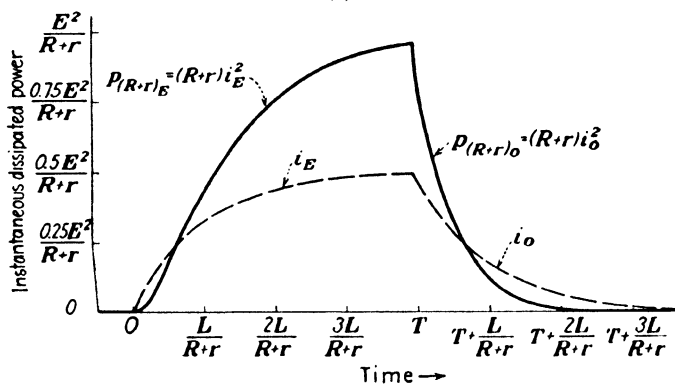
$$p_{R+r} = \frac{E^2}{R+r} \left[ e^{\frac{(R+r)T}{L}} - 1 \right]^2 e^{-\frac{2(R+r)t}{L}} \quad (65)$$

$$p_L = -\frac{E^2}{R+r} \left[ e^{\frac{(R+r)T}{L}} - 1 \right]^2 e^{-\frac{2(R+r)t}{L}} \quad (66)$$

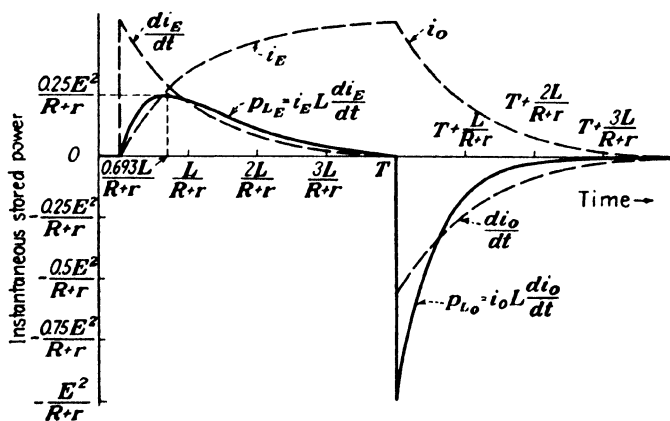
These equations are illustrated in Fig. 62 for a pulse width that is equal to four times the time constant. Curves of  $e_g$ ,  $i$ , and  $di/dt$  are also shown. They indicate that the generator



(a)



(b)



(c)

FIG. 62.—Behavior of instantaneous power in a basic  $RL$  network that is subjected to a rectangular-pulse voltage.

power is proportional to the product  $e_o i$ , that the resistance power is proportional to  $i^2$ , and that the inductance power is proportional to the product of  $i$  and  $di/dt$ .

*Power during Pulse.*—Power is delivered by the generator only during the time interval  $t = 0$  to  $t = T$ . See Fig. 62a. The instantaneous supplied power increases exponentially in accordance with a time constant equal to  $L/(R + r)$ . This power is being delivered to the resistance and inductance. The instantaneous power being delivered to  $(R + r)$  increases continuously with time, Fig. 62b, while the instantaneous power being delivered to  $L$ , Fig. 62c, reaches a maximum value of  $\frac{1}{4}E^2/(R + r)$  at a time equal to  $0.693L/(R + r)$  and then decreases.<sup>1</sup>

Inspection of Eqs. (62), (63), and (64) reveals that the instantaneous power supplied by the generator is always equal to the sum of the instantaneous power delivered to  $(R + r)$  and to  $L$ .

$$p_o \equiv p_{R+r} + p_L$$

*Power after Pulse.*—In the time interval from  $t = T$  on, the power delivered to  $L$  is released and delivered to  $(R + r)$ . Examination of the power equations for this time interval leads to the conclusion that  $p_L$  and  $p_{R+r}$  are always equal and opposite. The instantaneous power delivered by  $L$  to  $(R + r)$  decreases exponentially in accordance with a time constant equal to  $\frac{1}{2}L/(R + r)$ .

**17. Energy Relations.**—To investigate the energy relations refer to Table III where it is indicated that the power equations must be integrated to find the energy equations. The results of the integration are

#### DURING PULSE

$$w_o = EIt + LI^2[\epsilon^{-\frac{(R+r)t}{L}} - 1] \quad (67)$$

$$w_{R+r} = EIt + \frac{1}{2}LI^2[4\epsilon^{-\frac{(R+r)t}{L}} - \epsilon^{-\frac{2(R+r)t}{L}} - 3] \quad (68)$$

$$w_L = \frac{1}{2}LI^2[1 + \epsilon^{-\frac{2(R+r)t}{L}} - 2\epsilon^{-\frac{(R+r)t}{L}}] \quad (69)$$

<sup>1</sup> See Chap. III, p. 64. In this case  $p_L$  is of the same form as  $p_C$  in Chap. III.

## AFTER PULSE

$$w_g = 0$$

$$w_{R+r} = EIT$$

$$+ \frac{1}{2}LI^2 \left\{ 2 \left[ \epsilon^{-\frac{(R+r)T}{L}} - 1 \right] - \left[ \epsilon^{\frac{(R+r)T}{L}} - 1 \right]^2 \epsilon^{-\frac{2(R+r)t}{L}} \right\} \quad (70)$$

$$w_L = \frac{1}{2}LI^2 \left[ \epsilon^{\frac{(R+r)T}{L}} - 1 \right]^2 \epsilon^{-\frac{2(R+r)t}{L}} \quad (71)$$

$I$  = steady-state value of current =  $E/(R + r)$ . These equations are illustrated in Fig. 63 for a pulse width that is equal to four times the time constant.

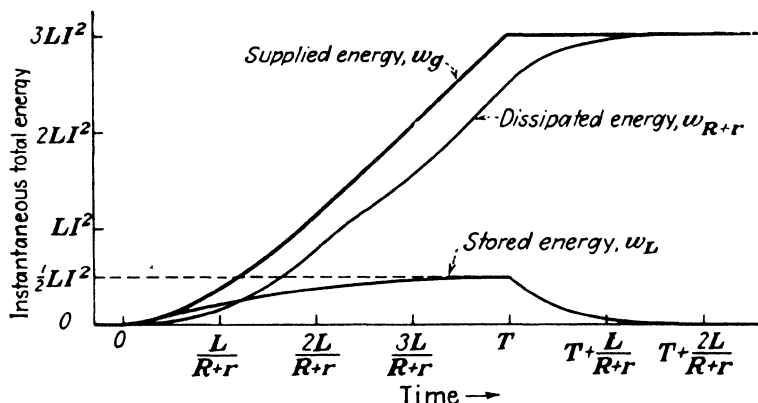


FIG. 63.—Behavior of instantaneous total energy in a basic  $RL$  network that is subjected to a rectangular-pulse voltage.

*Energy during Pulse.*—If the energy equations for  $w_L$  and  $w_{R+r}$  are added in the time interval  $t = 0$  to  $t = T$ , their sum is seen to be exactly equal to the energy equation for  $w_g$ . This means that the sum of the stored and dissipated energy must always equal the energy supplied by the generator.

If the pulse width is large compared with the time constant, then at the time  $t = T$  the energy equations become

$$\left. \begin{aligned} w_g &= EIT - LI^2 \\ w_{R+r} &= EIT - \frac{3}{2}LI^2 \\ w_L &= \frac{1}{2}LI^2 \end{aligned} \right\} T \gg \frac{L}{R+r}; \quad t = T$$

because both  $\epsilon^{-\frac{2(R+r)T}{L}}$  and  $\epsilon^{\frac{(R+r)T}{L}}$  are much less than 1 when  $T \gg L/(R + r)$ . This indicates that the energy supplied by the generator and the energy dissipated in  $(R + r)$  is continually

increasing during the pulse, but the energy stored in  $L$  reaches a maximum value of  $\frac{1}{2}LI^2$  and increases no further. Refer to Fig. 63.

When the pulse width is small compared with the time constant, very little energy is dissipated in  $(R + r)$  during the pulse, and most of it is stored in the magnetic field of  $L$ . This is shown clearly in Fig. 63 during the early portion of the pulse interval and explains why very little energy is delivered to the output terminals during the pulse when the network is used for "integration."

*Energy after Pulse.*—In the time interval  $t = T$  to  $t = \infty$  whatever energy has been stored in the magnetic field of the inductance must be dissipated in  $(R + r)$ . The amount of energy lost from the inductance is always exactly equal to the amount of energy dissipated in  $(R + r)$  during this time interval. This can be shown conveniently by rearranging Eq. (70).

$$w_{R+r} = EIT + LI^2 \left[ \epsilon^{-\frac{(R+r)T}{L}} - 1 \right] - \frac{1}{2}LI^2 \left[ \epsilon^{-\frac{(R+r)T}{L}} - 1 \right]^2 \epsilon^{-\frac{2(R+r)t}{L}} \quad (70a)$$

The first two terms are equal to the total energy supplied by the generator during the pulse interval. The last term is exactly equal to the energy stored in the magnetic field of the inductance, Eq. (71), except for sign. Therefore, the energy dissipated in  $(R + r)$  increases by exactly the same amount that the electromagnetic energy decreases. When  $t = \infty$ , both Eq. (71) and the last term in Eq. (70a) become zero, and all the energy delivered by the generator has been dissipated in  $(R + r)$ .

*Energy Summary.*—A summary of the important energy considerations can be made on the basis of Fig. 63. From this figure the following facts should now be evident:

1. The generator supplies energy only during the pulse interval.
2. The amount of energy supplied by the generator to the network can increase without limit as the pulse width increases.
3. The sum of the dissipated and stored energy is always equal to the energy supplied by the generator.
4. The maximum energy that can be stored in the inductance is  $\frac{1}{2}LI^2$  where  $I$  is the steady-state value of current equal to  $E/(R + r)$ .

5. During a pulse that is of short duration compared with the time constant more energy is stored than is dissipated.

6. All of the generator energy is eventually dissipated in the resistance.

### RL NETWORK WITH BOTH RESISTANCE AND INDUCTANCE ACROSS OUTPUT

If the resistance inherent in a "practical" inductor is not negligible, it will have an influence upon the pulse-response characteristic, especially when the output voltage is taken across the inductor. Such a situation is shown in Fig. 64 where

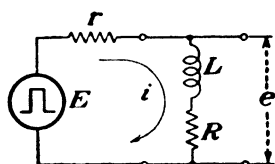


FIG. 64.—Series  $RL$  network with a rectangular-pulse generator.

$R$  represents the resistance contained in the inductor. It is tacitly assumed that an inductor that has an appreciable amount of resistance can be represented by the resistance of the coil in series with a "pure" inductance. This network can be analyzed quite readily because the equations for voltage across  $R$  and across  $L$  have already been developed for such a network.

**18. Equations for Output Pulse.**—The equations for voltage across  $R$  and across  $L$  in the network in Fig. 64 are

#### DURING PULSE

$$e_{R_s} = \frac{ER}{R+r} \left[ 1 - e^{-\frac{(R+r)t}{L}} \right] \quad (54)$$

$$e_{L_s} = E e^{-\frac{(R+r)t}{L}} \quad (59)$$

#### AFTER PULSE

$$e_{R_o} = \frac{ER}{R+r} \left[ e^{\frac{(R+r)T}{L}} - 1 \right] e^{-\frac{(R+r)t}{L}} \quad (57)$$

$$e_{L_o} = -E \left[ e^{\frac{(R+r)T}{L}} - 1 \right] e^{-\frac{(R+r)t}{L}} \quad (61)$$

Since the output voltage is the sum of the voltage across  $R$  and across  $L$ , then

$$e_E = e_{R_s} + e_{L_s} = \frac{E}{R+r} \left[ R + r e^{-\frac{(R+r)t}{L}} \right] \quad (72)$$

$$e_o = e_{R_o} + e_{L_o} = -\frac{Er}{R+r} \left[ e^{\frac{(R+r)T}{L}} - 1 \right] e^{-\frac{(R+r)t}{L}} \quad (73)$$

**19. Network Behavior.**—The behavior of the network in Fig. 64 is quite similar to that in Fig. 54 with a few notable exceptions. At  $t = 0$ , Eq. (72) indicates that the output voltage is  $E$ . This is to be expected because in any series  $RL$  network all the generator voltage will appear across the inductance at the instant the generator pulse arrives. After  $t = 0$ , the output voltage decreases in accordance with a time constant equal to  $L/(R + r)$ . The steady-state value of output voltage is  $Er/(R + r)$  and is reached when the voltage across the inductance becomes negligible.

If the pulse width is large compared with the time constant, then at  $t = T$  the output voltage equals the steady-state value. At the instant the generator pulse disappears, the output voltage changes instantly from a value  $Er/(R + r)$  to  $-Er/(R + r)$ . This can be seen from Eq. (73), which becomes

$$e_0 = -\frac{Er}{R + r} \epsilon^{-\frac{(R+r)(t-T)}{L}}$$

when  $T$  is so large compared with  $L/(R + r)$  that  $\epsilon^{\frac{(R+r)T}{L}} \gg 1$ . From the time  $T$  on, the output voltage approaches zero exponentially with a time constant equal to  $L/(R + r)$ .

To see what influence the relative values of  $R$  and  $r$  have upon the pulse-response characteristic, consider two extreme examples.

*Example 1.*  $R \ll r$ .—When  $R$  is negligibly small compared with  $r$ , then Eqs. (72) and (73) reduce to

$$e_K \approx E \epsilon^{-\frac{rt}{L}} \quad (72a)$$

$$e_0 \approx -E(\epsilon^{\frac{rT}{L}} - 1) \epsilon^{-\frac{rt}{L}} \quad (73a)$$

From these equations the following facts are evident:

1. The time constant is determined essentially by  $r$  and  $L$ .
2. The steady-state value of voltage during the generator pulse is essentially zero.
3. The output voltage can have a maximum negative value that is approximately equal to  $E$  if the generator pulse width is large compared with  $L/r$ .
4. The output voltage is of the same form as that of the network in Fig. 54. [Compare Eqs. (59) and (61) with Eqs. (72a) and (73a) when  $(R + r) \approx r$ .]

*Example 2.*  $R \gg r$ .—When the generator resistance is very small compared with  $R$ , then Eqs. (72) and (73) become

$$e_s \approx E \quad (72b)$$

$$e_o \approx 0 \quad (73b)$$

This shows that the output pulse is essentially the same as the generator pulse in shape and amplitude. Recall that the voltages across  $R$  and  $L$  considered individually are not the

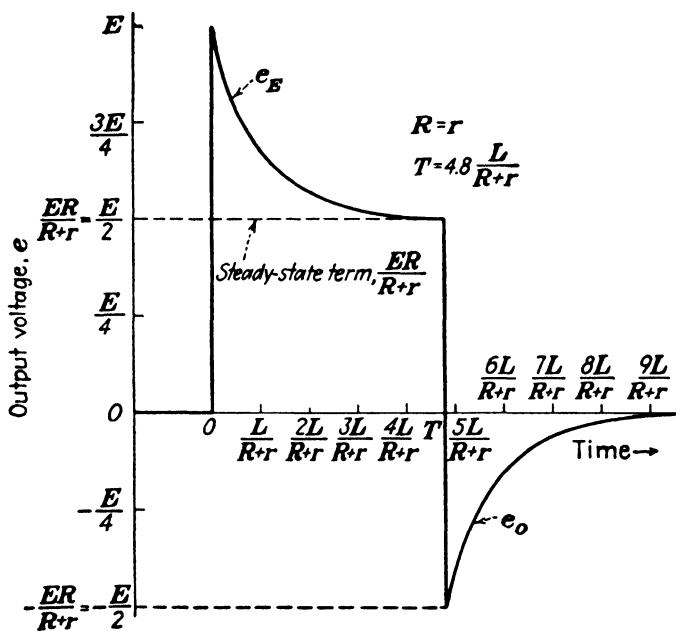


FIG. 65.—Output voltage of the network in Fig. 64 when  $R = r$  and  $T = 4.8L/(R + r)$ .

same as the generator voltage; nevertheless, their *sum* is a constant during the generator pulse and zero after the generator pulse. Refer to Fig. 36 for a graphical representation of this condition.

Figure 65 illustrates the output pulse of the network in Fig. 64 for a generator pulse that is equal to 4.8 times the time constant, and for  $R = r$ . The change in shape of this pulse for changes in relative values of  $R$  and  $r$  can be visualized by adjusting the steady-state value and most negative value of voltage in an appropriate manner. Output-pulse shapes for

other pulse widths can be deduced in a general way from those given for the network in Fig. 54.

### SUMMARY AND GENERALIZATION OF RESULTS

It is advantageous to reflect upon the three *RL* networks analyzed and to arrive at some generalities. The foregoing analyses can be manipulated into a form applicable to *any* series network that contains resistance and inductance only. Before the general output-pulse equations are developed, recall that series inductors are additive.<sup>1</sup> Since this is the case, all inductors in a series network can be added (lumped) into a single inductance. Likewise, all series resistors can be combined into a single equivalent resistance.

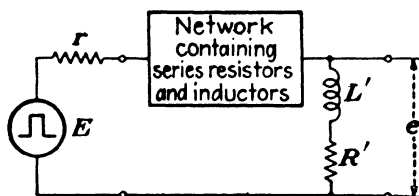


FIG. 66.—A general series *RL* network with a rectangular-pulse generator.

**20. General Output-voltage Equations.**—The network in Fig. 66 is a general network that includes not only the three networks already analyzed, but also *any* series *RL* network. Obviously, if the output pulse appears across  $R'$  only, then  $L'$  can be set equal to zero, and vice versa. Moreover,  $R'$  and  $L'$  can be the equivalent parameters of resistors and inductors that might be connected across the output.

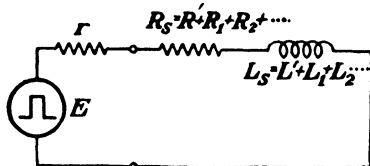


FIG. 67.—Equivalent series network for that in Fig. 66.

To make clear the derivation of the output-voltage equations that apply to this general network, the series equivalent network of Fig. 66 is shown in Fig. 67 where  $R'$  and  $L'$  have been combined with the other parameters. The equations for current and rate of change of current given by Eqs. (53), (56), (58), and (60) will apply equally well to this network because no restrictions

<sup>1</sup> Chap. I, p. 14.

were placed upon the values of  $R$  and  $L$  in their derivation. These equations written in terms of the general values of resistance and inductance are

## DURING PULSE

$$i_E = \frac{E}{R_s + r} \left[ 1 - e^{-\frac{(R_s + r)t}{L_s}} \right] \quad (53')$$

$$\frac{di_E}{dt} = \frac{E}{L_s} e^{-\frac{(R_s + r)t}{L_s}} \quad (58')$$

## AFTER PULSE

$$i_0 = \frac{E}{R_s + r} \left[ e^{\frac{(R_s + r)T}{L_s}} - 1 \right] e^{-\frac{(R_s + r)t}{L_s}} \quad (56')$$

$$\frac{di_0}{dt} = -\frac{E}{L_s} \left[ e^{\frac{(R_s + r)T}{L_s}} - 1 \right] e^{-\frac{(R_s + r)t}{L_s}} \quad (60')$$

The general output voltage will be the algebraic sum of the voltage across  $R'$  and across  $L'$ . The voltages to be summed are obtained by multiplying the current equations by  $R'$  and the rate of change of current equations by  $L'$ .

## DURING PULSE

$$e_{R'} = \frac{ER'}{R_s + r} \left[ 1 - e^{-\frac{(R_s + r)t}{L_s}} \right]$$

$$e_{L'} = \frac{EL'}{L_s} e^{-\frac{(R_s + r)t}{L_s}}$$

## AFTER PULSE

$$e_{R'_0} = \frac{ER'}{R_s + r} \left[ e^{\frac{(R_s + r)T}{L_s}} - 1 \right] e^{-\frac{(R_s + r)t}{L_s}}$$

$$e_{L'_0} = -\frac{EL'}{L_s} \left[ e^{\frac{(R_s + r)T}{L_s}} - 1 \right] e^{-\frac{(R_s + r)t}{L_s}}$$

Addition of these voltages yields the general output equations.

$$e_E = \frac{E}{R_s + r} \left\{ R' + \left[ \frac{L'}{L_s} (R_s + r) - R' \right] e^{-\frac{(R_s + r)t}{L_s}} \right\} \quad (74)$$

$$e_0 = E \left[ e^{\frac{(R_s + r)T}{L_s}} - 1 \right] \left[ \frac{R'}{R_s + r} - \frac{L'}{L_s} \right] e^{-\frac{(R_s + r)t}{L_s}} \quad (75)$$

These equations represent in mathematical form a complete extension of the output-voltage analysis of this chapter and contain all of the solutions previously found as well as the solu-

tion for any series network containing resistance and inductance only.

As an illustration of the use of these general equations, return to the network in Fig. 46. This simple network can be represented in terms of the general *RL* network if the following is true:

$$\begin{aligned} R_s &= R' = R \\ L' &= 0 \\ L_s &= L \end{aligned}$$

Substitute these values into the general equations.

$$\begin{aligned} e_E &= \frac{E}{R+r} \left\{ R + \left[ \frac{0}{L} (R+r) - R \right] \epsilon^{-\frac{(R+r)t}{L}} \right\} \\ &= \frac{E}{R+r} [R - R\epsilon^{-\frac{(R+r)t}{L}}] \\ &= \frac{ER}{R+r} [1 - \epsilon^{-\frac{(R+r)t}{L}}] \end{aligned} \quad (74a)$$

$$\begin{aligned} e_0 &= E[\epsilon^{\frac{(R+r)T}{L}} - 1] \left[ \frac{R}{R+r} - \frac{0}{L} \right] \epsilon^{-\frac{(R+r)t}{L}} \\ &= \frac{ER}{R+r} [\epsilon^{\frac{(R+r)T}{L}} - 1] \epsilon^{-\frac{(R+r)t}{L}} \end{aligned} \quad (75a)$$

Equations (74a) and (75a) are the same as Eqs. (54) and (57).

**21. Conclusion.**—Before concluding this chapter the striking similarity between *RL* and *RC* networks should be emphasized. Both networks are capable of giving output pulses that approach the derivative or integral of the generator pulse, or that approach an exact reproduction of the generator pulse. To sharpen the comparison between the two types, the following statement is useful: In a series *RL* or *RC* network where the output pulse appears across one element only, an identical pulse output can be obtained with either an *RL* or an *RC* network provided the network parameters are properly chosen. To verify this, compare some of the output-pulse shapes in this chapter with those in Chap. III.

The major pulse-response characteristics of series *RL* networks have been found, which, when combined with those of series *RC* networks, represent a good deal of information. To complete the analysis of series networks it is necessary to consider

the case where all three parameters are present. This is done in Chap. V.

### Problems

**Prob. 1.** The parameters of the network in Fig. 46 have the following values:  $E = 120$  volts,  $T = 75$  microseconds,  $r = 1,000$  ohms,  $R = 5,000$  ohms, and  $L = 0.3$  henry.

- What is the instantaneous current at  $t = 85$  microseconds?
- What is the maximum output voltage?
- "Pulse width" is sometimes defined as the time between the two instantaneous values of voltage that are  $1/\sqrt{2}$  times the maximum voltage. What is the pulse width of the output voltage on the basis of this definition? What is the pulse width of the generator voltage on the basis of this definition?

**Prob. 2.** The parameters of the network in Fig. 54 have the following values:  $E = 30$  volts,  $T = 230$  microseconds,  $r = 100$  ohms,  $R = 900$  ohms, and  $L = 0.1$  henry.

- What is the ratio of the maximum output voltage to the positive output voltage at  $t = T$ ?
- What is the minimum value of output voltage?
- How much energy is stored in the magnetic field of the inductance at  $t = 50$  microseconds?

**Prob. 3.** A series  $RL$  network has a time constant equal to 0.01 sec. The voltage across the inductance at the instant a 0.1-sec. rectangular pulse is applied is 100 volts.

- What is the generator-pulse voltage?
- Evaluate the following output-voltage ratios:

$$\frac{(e_E)_{t=0}}{(e_E)_{t=0.01}} \qquad \frac{(e_E)_{t=0.01}}{(e_E)_{t=0.02}} \qquad \frac{(e_E)_{t=0.02}}{(e_E)_{t=0.03}} \qquad \frac{(e_E)_{t=0.08}}{(e_E)_{t=0.09}}$$

- What relationship do the ratios in *b* have to  $e$ ?

**Prob. 4.** A rectangular-pulse generator has the following properties:  $E = 100$  volts,  $T = 75$  microseconds, and  $r = 1,000$  ohms. A 4,000-ohm resistor and a capacitor  $C$  are connected in series across the generator terminals, and the output voltage is taken across  $C$ . What value of  $C$  is required to have an output voltage that is identical to that in Prob. 1?

**Prob. 5.** An  $RL$  network has a time constant equal to 0.0001 sec. The inductor has an inductance of 0.1 henry and contains an inherent resistance of 100 ohms. The generator-pulse voltage is 80 volts, and the output voltage is taken across the inductor (which contains resistance).

- What is the maximum output voltage?
- What is the steady-state value of output voltage during the generator pulse?
- What pulse width is required to have the positive output voltage at  $t = T$  equal in magnitude to the negative output voltage at  $t = T$ ?
- What is the answer to *c* if the 100-ohm resistance of the inductor is neglected completely?

**Prob. 6.** It is desired to produce a 10-microsecond "rectangular" voltage across an inductor of 0.04 henry. The amplitude of this rectangular voltage must not decrease by more than 3 per cent during the generator pulse. What is the maximum tolerable value of generator internal resistance that will result in the desired output voltage? (Assume the inductor has negligible resistance.)

**Prob. 7.** The "time delay" between generator pulse and output pulse is sometimes defined as the difference in time between  $t = 0$  and the time when the output pulse has reached one-half its maximum value. What is the "time delay" in the network of Prob. 1?

**Prob. 8.** A rectangular-pulse generator has a negligible internal resistance compared with the resistance in a series *RL* network to which it is connected. The network time constant is 0.005 sec. The voltage across the resistance reaches 90 per cent of the generator voltage in a time equal to  $\frac{1}{10}$  of the generator-pulse width. What is the generator-pulse width?

## CHAPTER V

### **SERIES NETWORKS CONTAINING RESISTANCE, INDUCTANCE, AND CAPACITANCE**

The networks analyzed so far have been restricted to include either inductance or capacitance but not both. Such restricted networks occur very often, and the foregoing analysis is useful to predict their behavior. Nevertheless, there are many cases where  $R$ ,  $L$ , and  $C$  are all present. The purpose of this chapter is to investigate the pulse-response characteristics of series networks that contain resistance, inductance, and capacitance.

Before proceeding with the detailed analysis, it is well to anticipate and evaluate some of the difficulties that will arise. The very first problem to be encountered appears when the differential equation is written. All of the networks dealt with so far have given rise to linear, first-order, first-degree equations that have had simple exponential solutions. The differential equation that arises here, however, is a linear, *second-order*, first-degree equation. While its solution can be obtained in a straightforward manner, it is more complicated in form than that of the first-order type. An additional complicating factor arises from the fact that the roots of the auxiliary equation, which largely determine the form of the output pulse, are dependent upon the relative values of resistance, inductance, and capacitance. Furthermore, these roots, as well as the constants of integration, are fairly lengthy algebraic expressions that are awkward to handle.

To minimize the rather trivial complications arising from lengthy and awkward algebraic expressions, substitutions are made. But there is no way to avoid the fact that this type of network is more complex than those handled previously. The fundamental reason for the complexity can be viewed on an energy basis. In this type of network there are two different elements that are capable of storing energy: inductance and capacitance. The possibility of a momentary transfer of energy

TABLE IV.—COMPARISON OF SERIES NETWORKS

	Series networks restricted to contain $R$ and $L$ or $C$	Series networks containing $R$ , $L$ , and $C$
Differential equation	Linear, first-order, first-degree with constant coefficients	Linear, second-order, first-degree with constant coefficients
Solution of differential equation	Simple exponential; contains transient and steady-state terms	More complicated than simple exponential; contains transient and steady-state terms
Constants of integration	Simple algebraic expressions involving $R$ and $L$ or $C$	Cumbersome algebraic expressions involving $R$ , $L$ , and $C$
Form of solution	Independent of relative values of $R$ and $L$ or $C$	Dependent upon relative values of $R$ , $L$ , and $C$ . There are three separate cases
Physical interpretation	Straightforward	Rather involved
Energy considerations	Only one element can store and release electrical energy	Both $L$ and $C$ can store and release electrical energy and can exchange energy
Output pulse shape	Can approach the shape of the generator pulse or can approach the integral or differential of the generator pulse	Can approach the shape of the generator pulse, can approach the integral or differential of the generator pulse, or can be a variety of complicated shapes some of which bear no resemblance to the generator pulse
Fundamental principles upon which the pulse-response characteristics are based	Both types of network have this in common	

and also a continued exchange of energy between the inductance and the capacitance, even long after the generator pulse disappears, must be recognized. Equations capable of describing charges and currents that account for this transfer of energy are necessarily complicated.

This preliminary discussion should indicate that the results will lack simplicity in their mathematical expression. However, they will contain the desired pulse-response information. A comparison of this type of network with the restricted type is made in Table IV.

### BASIC *RLC* NETWORK WITH CAPACITANCE ACROSS OUTPUT

The first specific network to be analyzed is shown in Fig. 68. The generator pulse is perfectly rectangular and has a constant value  $E$  during its duration  $T$ .

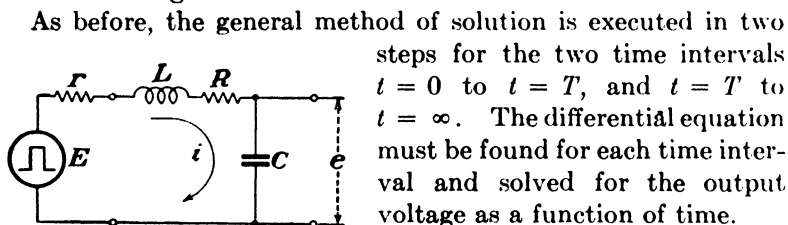


FIG. 68.—Series *RLC* network with a rectangular-pulse generator. Capacitance output.

As before, the general method of solution is executed in two steps for the two time intervals  $t = 0$  to  $t = T$ , and  $t = T$  to  $t = \infty$ . The differential equation must be found for each time interval and solved for the output voltage as a function of time.

**1. Equations for Output Pulse;  $t = 0$  to  $t = T$ .**—Application of Kirchhoff's laws to the network in

Fig. 68 leads to the following differential equation for the time interval during which the generator pulse exists:

$$E = ri_E + L \frac{di_E}{dt} + Ri_E + \frac{q_E}{C} \quad (76)$$

The ultimate aim is to solve this differential equation for  $q_E$  as a function of time and to divide  $q_E$  by  $C$  to obtain the output voltage.

Evidently  $i_E$  and  $di_E/dt$  must be written in terms of  $q_E$ . Since  $i_E = dq_E/dt$ , then  $di_E/dt = d^2q_E/dt^2$ . Substitution of these fundamental relationships into Eq. (76) yields

$$L \frac{d^2q_E}{dt^2} + (R + r) \frac{dq_E}{dt} + \frac{q_E}{C} = E \quad (77)$$

This is a linear, second-order, first-degree equation that has a solution consisting of the sum of the particular integral and

the complementary function. Recall that the particular integral corresponds to the steady-state term and the complementary function corresponds to the transient term.<sup>1</sup> Since it is not possible to solve Eq. (77) by separation of variables, it becomes necessary to evaluate the particular integral and complementary function individually.

*Particular Integral.*—To obtain the particular integral, or steady-state term, one can reason that  $q_E$  will be constant when the steady state is reached. If  $q_E$  is constant, then both  $dq_E/dt$  and  $d^2q_E/dt^2$  are zero; thus Eq. (77) becomes

$$q_E = CE$$

To verify the fact that  $q_E = CE$  is a solution of Eq. (77),  $CE$  can be substituted for  $q_E$ , resulting in an identity.

*Complementary Function.*—It was disclosed in Chap. II that the complementary function, or transient term, is of the form

$$q'_E = K\epsilon^{-At} \quad (20)$$

where  $A$  and  $K$  are constants that have values determined by the network parameters and initial conditions respectively. The complementary function will be the complete solution of the differential equation when  $E = 0$ . If this assumed form of  $q'_E$  is differentiated and substituted into Eq. (77), setting  $E = 0$ , the following equation results:

$$LA^2K\epsilon^{-At} - (R + r)AK\epsilon^{-At} + \frac{K}{C}\epsilon^{-At} = 0$$

Factor out  $K\epsilon^{-At}$ .

$$K\epsilon^{-At} \left[ A^2 - \left( \frac{R + r}{L} \right) A + \frac{1}{LC} \right] = 0$$

The auxiliary equation is

$$A^2 - \left( \frac{R + r}{L} \right) A + \frac{1}{LC} = 0$$

It is convenient to define two symbols at this point:

$$M \equiv \frac{R + r}{2L}; \quad N \equiv \frac{1}{LC}$$

<sup>1</sup> Chap. II, p. 26.

where  $M$  and  $N$  will be called *secondary* network parameters.  $M$  and  $N$  will be used throughout this chapter rather than  $r$ ,  $R$ ,  $L$ , and  $C$  in order to simplify the notation. The dimension of  $M$  is  $1/(\text{time})$  and of  $N$  is  $1/(\text{time})^2$ .

There are two values of  $A$  that satisfy the auxiliary equation. They are

$$\begin{aligned} A_1 &= M + \sqrt{M^2 - N} \\ A_2 &= M - \sqrt{M^2 - N} \end{aligned}$$

If these two values of  $A$  are not equal, the complementary function will be<sup>1</sup>

$$q'_E = K_1 e^{-A_1 t} + K_2 e^{-A_2 t} \quad (78)$$

where  $K_1$  and  $K_2$  are constants to be evaluated from the initial conditions. If  $A_1 = A_2 = M$ , which occurs when  $M^2 = N$ , then the complementary function is

$$q'_E = (K_3 - K_4 t) e^{-Mt} \quad (79)$$

*Output Voltage.*—Addition of the steady-state and transient terms results in the complete solution for instantaneous charge.

$$\text{If } A_1 \neq A_2 \quad q_E = CE + K_1 e^{-A_1 t} + K_2 e^{-A_2 t} \quad (80)$$

$$\text{If } A_1 = A_2 = M \quad q_E = CE + (K_3 - K_4 t) e^{-Mt} \quad (81)$$

The output voltage is  $q_E/C$ .

$$\text{If } A_1 \neq A_2 \quad e_E = E + \frac{K_1}{C} e^{-A_1 t} + \frac{K_2}{C} e^{-A_2 t} \quad (82)$$

$$\text{If } A_1 = A_2 = M \quad e_E = E + \left( \frac{K_3 - K_4 t}{C} \right) e^{-Mt} \quad (83)$$

The transient and steady-state terms are clearly distinguishable in these equations.

The constants  $K_1$ ,  $K_2$ ,  $K_3$ , and  $K_4$  must be evaluated from the initial conditions to obtain specific solutions for the output voltage. There are three cases that must be considered to include all possible forms of the output voltage. The three cases are:  $A_1 \neq A_2$  where  $A_1$  and  $A_2$  are real,  $A_1 \neq A_2$  where  $A_1$  and  $A_2$  are complex, and  $A_1 = A_2 = M$ .

**Case 1.**  $M^2 > N$  or  $(R + r) > 2 \sqrt{L/C}$ .—This is called the *overdamped* case for reasons that will become evident. In this case  $A_1$  and  $A_2$  are real and unequal, and Eqs. (80) and (82)

<sup>1</sup> Chap. II, p. 29.

are applicable. It is necessary to know the value of charge and current in the network in Fig. 68 at the time  $t = 0$  to evaluate  $K_1$  and  $K_2$ . The presence of inductance in the network opposes any change in current while the capacitance offers no opposition to current at  $t = 0$ . Therefore, at  $t = 0$  the current must be zero because the current cannot change instantly. Consequently, the charge on  $C$  must also be zero at  $t = 0$  because the flow of charge is current.

Inserting  $q_E = 0$  and  $t = 0$  into Eq. (80) results in one equation involving  $K_1$  and  $K_2$ .

$$K_1 + K_2 = -CE$$

A second equation results when Eq. (80) is differentiated

$$i_E = \frac{dq_E}{dt} = -A_1 K_1 e^{-A_1 t} - A_2 K_2 e^{-A_2 t}$$

and the condition  $i_E = 0$  at  $t = 0$  is inserted.

$$0 = -A_1 K_1 - A_2 K_2$$

Solve these two simultaneous equations for  $K_1$  and  $K_2$ .

$$K_1 = CE \frac{A_2}{A_1 - A_2}$$

$$K_2 = -CE \frac{A_1}{A_1 - A_2}$$

Substitute the values of  $K_1$  and  $K_2$  into Eq. (82).

$$e_E = E - E \left( \frac{A_1}{A_1 - A_2} e^{-A_1 t} - \frac{A_2}{A_1 - A_2} e^{-A_2 t} \right) \quad (82a)$$

When the values of  $A_1$  and  $A_2$  in terms of  $M$  and  $N$  are substituted into Eq. (82a), it becomes

$$e_E = E - \frac{E}{2\sqrt{M^2 - N}} \{ [M + \sqrt{M^2 - N}] e^{-(M - \sqrt{M^2 - N})t} \\ - [M - \sqrt{M^2 - N}] e^{-(M + \sqrt{M^2 - N})t} \}$$

Factor out  $e^{-Mt}$  and group terms.

$$e_E = E - \frac{E e^{-Mt}}{\sqrt{M^2 - N}} \left[ M \left( \frac{e^{\sqrt{M^2 - N}t} - e^{-\sqrt{M^2 - N}t}}{2} \right) \right. \\ \left. + \sqrt{M^2 - N} \left( \frac{e^{\sqrt{M^2 - N}t} + e^{-\sqrt{M^2 - N}t}}{2} \right) \right] \quad (82b)$$

This is a perfectly valid solution for the output voltage, but considerable simplification is possible if hyperbolic functions are introduced. The first step is to eliminate the exponential terms in the brackets by recognizing that the first term can be replaced by a hyperbolic sine, and that the second term can be replaced by a hyperbolic cosine.

$$e_E = E - \frac{E\epsilon^{-Mt}}{\sqrt{M^2 - N}} \\ (M \sinh \sqrt{M^2 - N} t + \sqrt{M^2 - N} \cosh \sqrt{M^2 - N} t)$$

Now recall from Chap. II the identity

$$P \sinh At + S \cosh At \equiv \sqrt{P^2 - S^2} \sinh \left( At + \tanh^{-1} \frac{S}{P} \right)$$

Therefore,

$$e_E = E - \frac{E\epsilon^{-Mt}}{\sqrt{M^2 - N}} \\ \left[ \sqrt{M^2 - M^2 + N} \sinh \left( \sqrt{M^2 - N} t + \tanh^{-1} \frac{\sqrt{M^2 - N}}{M} \right) \right]$$

This can be written

$$e_E = E - \frac{E\epsilon^{-Mt}}{\sqrt{(M^2 - N)/N}} \sinh (\sqrt{M^2 - N} t + \alpha) \quad (84)$$

where  $\alpha = \tanh^{-1} \frac{\sqrt{M^2 - N}}{M} = \text{constant}$ . Equation (84) is the complete solution for output voltage in terms of the secondary network parameters for the overdamped case and during the generator pulse.

**Case 2.**  $M^2 < N$  or  $(R + r) < 2\sqrt{L/C}$ .—This is called the *underdamped* or *oscillatory* case because the output voltage is oscillatory. In this case  $A_1$  and  $A_2$  are imaginary and unequal because  $\sqrt{M^2 - N}$  is imaginary. Equations (80) and (82) are again applicable. Equation (82b), which is basically the same as Eq. (82), becomes

$$e_E = E - \frac{E\epsilon^{-Mt}}{\sqrt{N - M^2}} \left[ M \left( \frac{\epsilon^{j\sqrt{N - M^2}t} - \epsilon^{-j\sqrt{N - M^2}t}}{2j} \right) \right. \\ \left. + \sqrt{N - M^2} \left( \frac{\epsilon^{j\sqrt{N - M^2}t} + \epsilon^{-j\sqrt{N - M^2}t}}{2} \right) \right]$$

because  $\sqrt{M^2 - N} = \sqrt{-(N - M^2)} = j\sqrt{N - M^2}$ .

$\sqrt{N - M^2}$  is a real number when  $M^2 < N$ . Replace the terms in the brackets by hyperbolic functions.

$$e_E = E - \frac{E\epsilon^{-Mt}}{\sqrt{N - M^2}} \left( M \frac{\sinh j \sqrt{N - M^2} t}{j} + \sqrt{N - M^2} \cosh j \sqrt{N - M^2} t \right)$$

As was shown in Chap. II, hyperbolic functions of imaginary numbers are related to trigonometric functions.

$$\sinh jAt = j \sin At; \quad \cosh jAt = \cos At$$

Consequently,

$$e_E = E - \frac{E\epsilon^{-Mt}}{\sqrt{N - M^2}} (M \sin \sqrt{N - M^2} t + \sqrt{N - M^2} \cos \sqrt{N - M^2} t)$$

Recall the trigonometric identity introduced in Chap. II:

$$P \sin At + S \cos At \equiv \sqrt{P^2 + S^2} \sin \left( At + \tan^{-1} \frac{S}{P} \right)$$

The output voltage therefore becomes

$$e_E = E - \frac{E\epsilon^{-Mt}}{\sqrt{N - M^2}} \left[ \sqrt{M^2 + N - M^2} \sin \left( \sqrt{N - M^2} t + \tan^{-1} \frac{\sqrt{N - M^2}}{M} \right) \right]$$

This can be written

$$e_E = E - \frac{E\epsilon^{-Mt}}{\sqrt{(N - M^2)/N}} \sin (\sqrt{N - M^2} t + \beta) \quad (85)$$

where  $\beta = \tan^{-1} \frac{\sqrt{N - M^2}}{M} = \text{constant}$ . Equation (85) is the complete solution for output voltage in terms of  $M$  and  $N$  for the oscillatory case and during the generator pulse.

**Case 3.**  $M^2 = N$  or  $(R + r) = 2\sqrt{L/C}$ .—This is called the *critically damped* case because it is the borderline case between the oscillatory and overdamped cases. Here  $A_1 = A_2 = M$ , and Eqs. (81) and (83) are applicable.

The constants  $K_3$  and  $K_4$  can be evaluated if the initial conditions  $q_E = 0$  and  $i_E = 0$  at  $t = 0$  are utilized. Insert  $q_E = 0$  and  $t = 0$  into Eq. (81) to find the value of  $K_3$ .

$$K_3 = -CE$$

Differentiate Eq. (81).

$$\frac{dq_E}{dt} = i_E = -MK_3\epsilon^{-Mt} + MK_4t\epsilon^{-Mt} - K_4\epsilon^{-Mt}$$

Insert the condition  $i_E = 0$  at  $t = 0$  to obtain the value of  $K_4$ .

$$K_4 = -MK_3 = MCE$$

Substitution of these values of  $K_3$  and  $K_4$  into Eq. (83) yields the equation for the output voltage.

$$e_E = E - E(1 + Mt)\epsilon^{-Mt} \quad (86)$$

This equation is the complete solution for the output voltage in terms of  $M$  for the critically damped case and during the generator pulse.

**2. Network Behavior;  $t = 0$  to  $t = T$ .**—Before continuing the analysis it is wise to attach some physical significance to the three output-voltage equations obtained thus far, and to examine the behavior of the network in Fig. 68 for the three cases.

**Case 1. Overdamped.**—In this case Eq. (84) describes the output voltage. Figure 69 is a graphical representation of Eq. (84) for a fixed pulse width and amplitude and two values of  $M^2$ , each of which is greater than  $N$ . The voltage is seen to rise slowly at first, then more rapidly, and then more slowly to approach the steady-state value  $E$ . Of course the charge on the capacitor is accumulating in a similar manner. The point of inflection on each curve, which is the point of maximum slope, corresponds to the condition of maximum current flow. This point will be treated in detail later.

Inspection of Eq. (84) bears out this output-voltage behavior. For illustration, consider the case where  $M^2 = 2N$ . If  $M^2 = 2N$ , then

$$\begin{aligned} \sqrt{\frac{M^2 - N}{N}} &= \sqrt{\frac{M^2}{N} - 1} = \sqrt{\frac{2N}{N} - 1} = 1 \\ \sqrt{M^2 - N} t &= \sqrt{\frac{M^2 - N}{M^2}} Mt = \frac{Mt}{\sqrt{2}} = 0.707Mt \\ \alpha &= \tanh^{-1} \frac{\sqrt{M^2 - N}}{M} = \tanh^{-1} \frac{1}{\sqrt{2}} = 0.881 \end{aligned}$$

Equation (84) becomes

$$e_E = E - E\epsilon^{-Mt} \sinh (0.707Mt + 0.881)$$

At  $t = 0$ , the output voltage is zero.

$$(e_E)_{t=0} = E - E \sinh (0.881) = E - E = 0$$

For values of  $t$  which are small compared with  $M$ , the exponential term is close to unity and the transient term is governed mainly by the hyperbolic sine. Consequently, the initial increase of output voltage approximates the hyperbolic sine. However, as  $t$  becomes larger, the exponential term begins to take hold, so to speak. The fact that the exponential term approaches zero more rapidly than the hyperbolic sine term approaches infinity explains why the transient eventually diminishes to zero.

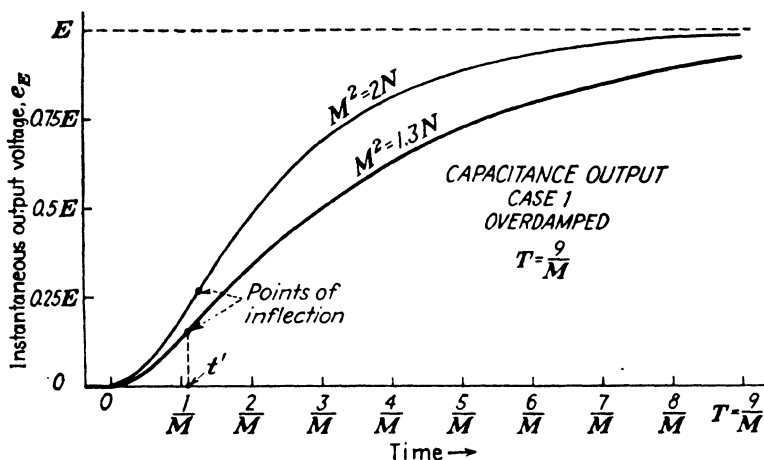


Fig. 69.- Output voltage of the network in Fig. 68 for the overdamped case and during the generator pulse.

Because the limit of an exponential times a hyperbolic sine can be either zero or infinity, it becomes necessary to investigate further. Consider the general case

$$e^{-at} \sinh (bt + c)$$

where  $a$ ,  $b$ , and  $c$  are constants. Convert to the exponential form.

$$\begin{aligned} e^{-at} \left( \frac{e^{bt+c} - e^{-bt-c}}{2} \right) &= \frac{1}{2} e^{-at} e^{bt+c} [1 - e^{-2(bt+c)}] \\ &= \frac{1}{2} e^{(b-a)t} e^c [1 - e^{-2(bt+c)}] \end{aligned}$$

Proceed to the limit as  $t$  approaches infinity.

$$\lim_{t \rightarrow \infty} e^{-at} \sinh(bt + c) = \frac{1}{2} e^{(b-a)\infty} e^c (1 - e^{-\infty}) = \frac{1}{2} e^c e^{(b-a)\infty}$$

Now if  $a < b$ , the limit is infinity, and if  $a > b$ , the limit is zero. In the case of Eq. (84),  $a = M$ ,  $b = \sqrt{M^2 - N}$ , and  $M^2 > N$ . Therefore,  $M > \sqrt{M^2 - N}$  or  $a > b$ , which proves that the limit is zero and that the transient term vanishes as  $t$  becomes large.

The output voltage reaches a steady-state value that is equal to the generator voltage when the transient becomes negligible.

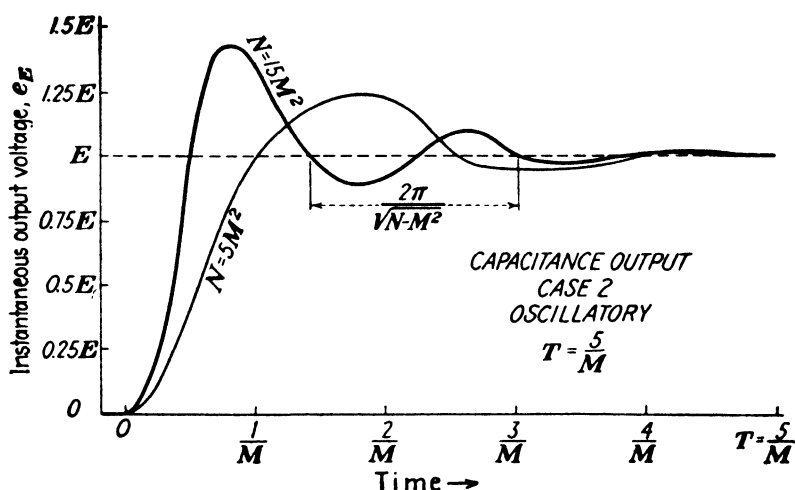


FIG. 70.—Output voltage of the network in Fig. 68 for the oscillatory case and during the generator pulse.

**Case 2. Oscillatory.**—In this case Eq. (85) applies. Figure 70 is a graphical representation of Eq. (85) for a fixed pulse width and amplitude and two values of  $M^2$ , each of which is less than  $N$ . The voltage is seen to rise slowly at first and then more rapidly. A maximum value that is greater than the generator-pulse voltage is reached and a damped oscillation occurs about the generator voltage  $E$ . The charge on the capacitor is going through a similar behavior. The point of inflection during the first rise of output voltage corresponds to a current maximum. The current behavior can be visualized by considering the slope of the output voltage, the slope being proportional to the current. Each time the output voltage is a maximum or a minimum, the

instantaneous current in the network is zero, and each time the slope of the output voltage is a maximum, point of inflection, the instantaneous current is a maximum. Notice that the points of inflection do not necessarily occur at times when the output voltage equals  $E$ .

An interpretation of Eq. (85) explains this output-voltage behavior. The steady-state value of output voltage is  $E$ , by inspection. To study the behavior of the transient term, it is useful to visualize a sine wave of constant amplitude that is multiplied by an exponential. In Fig. 71 the two constituents

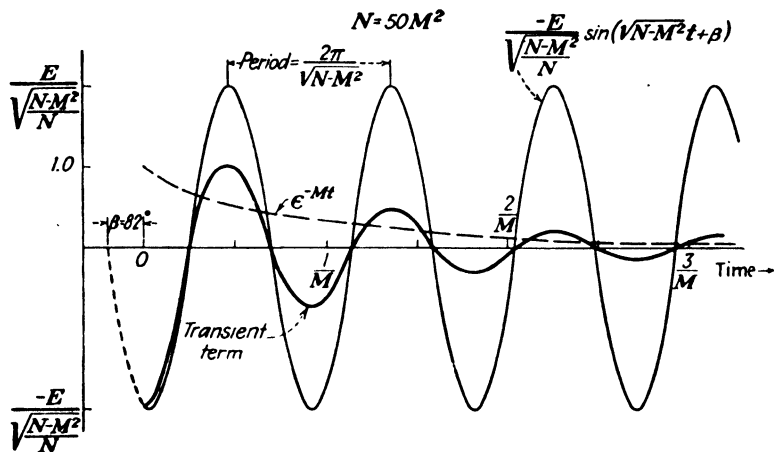


FIG. 71.—Graphical representation of the transient term in the oscillatory case as the product of an exponential and sinusoidal term.

of the transient term are shown along with their product. Thus it is seen that the transient term is a damped oscillation that diminishes to zero. Hence, the output voltage is oscillatory and attains the steady-state value  $E$  when the transient becomes negligible.

An inquiry into the period of the oscillation of output voltage can be made. The angular velocity of the sinusoidal term is  $\sqrt{N - M^2}$  and, since the angular velocity equals  $2\pi$  divided by the period, the period of oscillation is given by

$$\text{Period} = \frac{2\pi}{\sqrt{N - M^2}} \quad (87)$$

This expression becomes approximately  $2\pi/\sqrt{N} = 2\pi\sqrt{LC}$  if  $M^2$  is so small compared with  $N$  that  $\sqrt{N - M^2} \approx \sqrt{N}$ .

If  $M^2$  is zero, an impossible case to achieve but one that can be approached, Eq. (85) becomes

$$e_E = E - E \sin \left( \sqrt{N} t + \frac{\pi}{2} \right)$$

The period of oscillation is exactly equal to  $2\pi \sqrt{LC}$ , and the exponential term has vanished. This means that the output voltage oscillates indefinitely between the extremities of zero voltage and  $2E$  as illustrated in Fig. 72. This suggests that the network is capable of generating a sine wave that has a frequency determined by  $L$  and  $C$ .

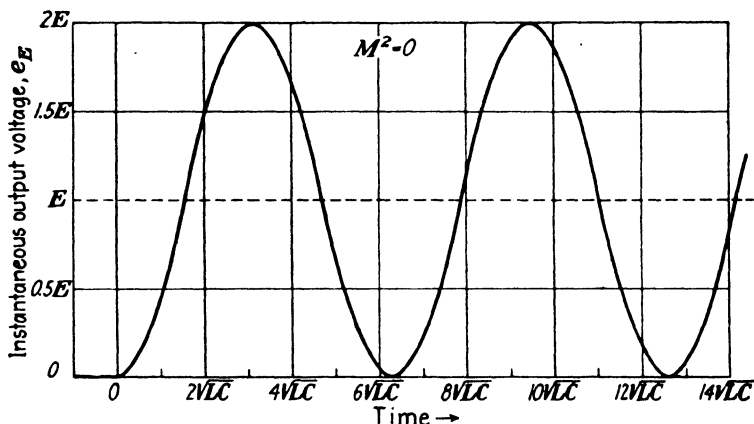


Fig. 72.—Output voltage of the network in Fig. 68 when  $(R + r) = 0$ . The period is equal to  $2\pi \sqrt{LC}$ .

As the value of  $M^2$  approaches  $N$ , the period approaches infinity, which is a trend toward the critically damped case.

**Case 3. Critically Damped.**—In this case Eq. (86) contains the output voltage information. Figure 73 shows the output voltage for the condition that  $M^2 = N$ . This case is very difficult to achieve in practice; nevertheless, it can be approached very closely. It is the borderline case between the overdamped and oscillatory cases; in other words, the network condition is such that oscillation is not quite possible.

A comparison of Fig. 73 with Fig. 69 reveals a similarity, but observe that the steady-state output voltage is attained more quickly in the critically damped case than in the overdamped case. This property of the critically damped condition

is often used to good advantage in applications where the duration of the transient must be minimized and where oscillation is undesired.

The transient term in Eq. (86) diminishes to zero as  $t$  becomes large because  $\epsilon^{-Mt}$  approaches zero more rapidly than  $(1 + Mt)$  approaches infinity. This can be demonstrated mathematically.

$$\begin{aligned}\lim_{t \rightarrow \infty} (1 + Mt)\epsilon^{-Mt} &= \lim_{t \rightarrow \infty} \epsilon^{-Mt} + \lim_{t \rightarrow \infty} Mt\epsilon^{-Mt} \\ &= 0 + \lim_{t \rightarrow \infty} \frac{Mt}{\epsilon^{Mt}}\end{aligned}$$

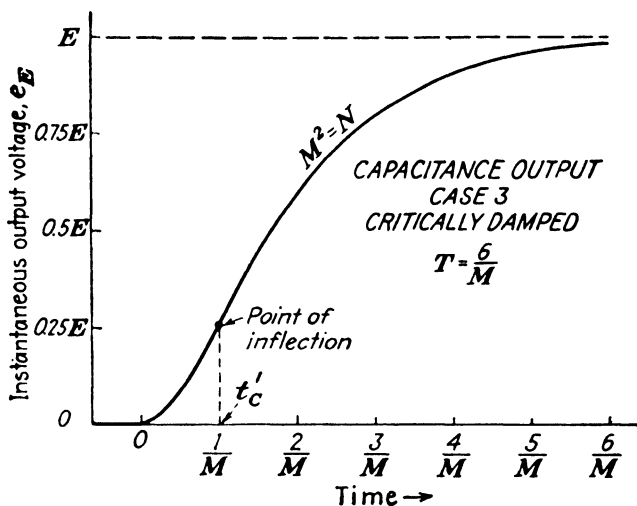


FIG. 73.—Output voltage of the network in Fig. 68 for the critically damped case and during the generator pulse.

In the limit this becomes indeterminate,  $\infty / \infty$ , so it is necessary to differentiate the numerator and the denominator. (The value of the new fraction as  $t$  approaches infinity is the limiting value of the original fraction.<sup>1</sup>) Differentiate the numerator and denominator separately and proceed to the limit.

$$\lim_{t \rightarrow \infty} \frac{Mt}{\epsilon^{Mt}} = \lim_{t \rightarrow \infty} \frac{M}{M\epsilon^{Mt}} = \frac{M}{\infty} = 0$$

Therefore, the transient term approaches zero, and the output

<sup>1</sup> Granville, Smith, and Longley, "Elements of the Differential and Integral Calculus," Ginn and Company, or any other standard calculus book.

voltage approaches a steady-state value equal to the generator voltage.

**3. Equations for Output Pulse ;  $t = T$  to  $t = \infty$ .**—Equations (84), (85), and (86) pertain to the time interval during which the generator pulse exists. To complete the solution one must consider the network of Fig. 68 after the generator pulse has disappeared. From the time  $T$  on, the differential equation that applies is Eq. (77) with the generator voltage set equal to zero. Write Eq. (77) in terms of  $M$  and  $N$ , and set  $E = 0$ .

$$\frac{d^2 q_0}{dt^2} + 2M \frac{dq_0}{dt} + Nq_0 = 0 \quad (77a)$$

*Particular Integral.*—To obtain the particular integral, or steady-state term, set  $d^2 q_0/dt^2 = dq_0/dt = 0$ . Equation (77a) indicates that the steady-state term is zero,  $q_0 = 0$ .

*Complementary Function.*—The complementary function must comprise the entire solution because the particular integral is zero. The solution of Eq. (77a) has been found previously. It is given by Eqs. (78) and (79) which are rewritten below with the constants  $K_1, K_2, K_3$ , and  $K_4$  replaced by  $K_5, K_6, K_7$ , and  $K_8$  because the conditions at  $t = 0$  and  $t = T$  are different.

$$\text{When } A_1 \neq A_2 \quad q_0 = K_5 e^{-A_1 t} + K_6 e^{-A_2 t} \quad (78a)$$

$$\text{When } A_1 = A_2 = M \quad q_0 = (K_7 - K_8 t) e^{-Mt} \quad (79a)$$

*Output Voltage.*—The output voltage during the time interval  $t = T$  to  $t = \infty$  is obtained by dividing Eqs. (78a) and (79a) by  $C$ .

$$\text{When } A_1 \neq A_2 \quad e_0 = \frac{K_5}{C} e^{-A_1 t} + \frac{K_6}{C} e^{-A_2 t} \quad (88)$$

$$\text{When } A_1 = A_2 = M \quad e_0 = \left( \frac{K_7 - K_8 t}{C} \right) e^{-Mt} \quad (89)$$

It can be seen clearly from these equations that the steady-state term is zero. Three cases must be considered to obtain specific solutions for the output voltage.

**Case 1.  $M^2 > N$ . Overdamped.**— $A_1$  and  $A_2$  are real and unequal. To evaluate  $K_5$  and  $K_6$  it is necessary to know the values of instantaneous charge and instantaneous current at  $t = T$ . These values are given by the following equations that

are obtained by setting  $t = T$  in Eq. (80) and in the derivative of Eq. (80).

$$q_{ET} = CE + K_1\epsilon^{-A_1T} + K_2\epsilon^{-A_2T} \quad (80a)$$

$$i_{ET} = \left( \frac{dq_E}{dt} \right)_T = -A_1K_1\epsilon^{-A_1T} - A_2K_2\epsilon^{-A_2T} \quad (80b)$$

Inserting the condition  $q_0 = q_{ET}$  at  $t = T$  into Eq. (78a) yields one equation for  $K_5$  and  $K_6$ .

$$q_{ET} = K_5\epsilon^{-A_1T} + K_6\epsilon^{-A_2T}$$

Another equation is obtained by differentiating Eq. (78a) and then setting  $i_0 = i_{ET}$  and  $t = T$ .

$$i_{ET} = -A_1K_5\epsilon^{-A_1T} - A_2K_6\epsilon^{-A_2T}$$

Solve these two simultaneous equations for  $K_5$  and  $K_6$ .

$$K_5 = \frac{CEA_2}{A_1 - A_2} (1 - \epsilon^{A_1T}) = K_1(1 - \epsilon^{A_1T})$$

$$K_6 = -\frac{CEA_1}{A_1 - A_2} (1 - \epsilon^{A_2T}) = K_2(1 - \epsilon^{A_2T})$$

Substitute the values of  $K_5$  and  $K_6$  into Eq. (88).

$$e_0 = \frac{E}{A_1 - A_2} [A_2(1 - \epsilon^{A_1T})\epsilon^{-A_1t} - A_1(1 - \epsilon^{A_2T})\epsilon^{-A_2t}] \quad (88a)$$

Expand this equation and group terms.

$$e_0 = E \left( \frac{A_2\epsilon^{-A_1t} - A_1\epsilon^{-A_2t}}{A_1 - A_2} \right) - E \left( \frac{A_2\epsilon^{-A_1(t-T)} - A_1\epsilon^{-A_2(t-T)}}{A_1 - A_2} \right) \quad (88b)$$

The first term in Eq. (88b) is identically the same as the transient term in Eq. (82a) which has been shown to become

$$-\frac{E\epsilon^{-Mt}}{\sqrt{(M^2 - N)}/N} \sinh(\sqrt{M^2 - N}t + \alpha)$$

when the values of  $A_1$  and  $A_2$  in terms of  $M$  and  $N$  were substituted. The second term in Eq. (88b) is the same as the first term except for sign and the replacement of  $t$  by  $(t - T)$ . So, by analogy, it becomes

$$\frac{E\epsilon^{-M(t-T)}}{\sqrt{(M^2 - N)}/N} \sinh[\sqrt{M^2 - N}(t - T) + \alpha]$$

Thus the output voltage in terms of the secondary parameters and pulse width for the overdamped case and after the generator pulse disappears is

$$e_0 = \frac{E\epsilon^{-Mt}}{\sqrt{(M^2 - N)/N}} \{ \epsilon^{Mt} \sinh [\sqrt{M^2 - N} (t - T) + \alpha] - \sinh (\sqrt{M^2 - N} t + \alpha) \} \quad (90)$$

When  $T$  is very large compared with  $1/M$ ,  $\epsilon^{Mt}$  is extremely large and the output-voltage equation becomes approximately

$$e_0 \approx \frac{E\epsilon^{-M(t-T)}}{\sqrt{(M^2 - N)/N}} \sinh [\sqrt{M^2 - N} (t - T) + \alpha] \quad (90a)$$

**Case 2.**  $M^2 < N$ . *Oscillatory.*— $A_1$  and  $A_2$  are imaginary and unequal. Under this condition the first term in Eq. (88b) becomes

$$- \frac{E\epsilon^{-Mt}}{\sqrt{(N - M^2)/N}} \sin (\sqrt{N - M^2} t + \beta)$$

which is the same as the transient term in Eq. (85). The second term in Eq. (88b), by analogy with the first term, becomes

$$\frac{E\epsilon^{-M(t-T)}}{\sqrt{(N - M^2)/N}} \sin [\sqrt{N - M^2} (t - T) + \beta]$$

Therefore, the output-voltage equation is

$$e_0 = \frac{E\epsilon^{-Mt}}{\sqrt{(N - M^2)/N}} \{ \epsilon^{Mt} \sin [\sqrt{N - M^2} (t - T) + \beta] - \sin (\sqrt{N - M^2} t + \beta) \} \quad (91)$$

This equation is the complete solution for the output voltage in terms of the secondary parameters and the pulse width for the oscillatory case and after the generator pulse disappears.

When the pulse width is very large compared with  $1/M$ , then  $\epsilon^{Mt}$  is extremely large, and the output-voltage equation becomes approximately

$$e_0 \approx \frac{E\epsilon^{-M(t-T)}}{\sqrt{(N - M^2)/N}} \sin [\sqrt{N - M^2} (t - T) + \beta] \quad (91a)$$

**Case 3.**  $M^2 = N$ . *Critically Damped.*— $A_1$  and  $A_2$  are equal, and equal to  $M$ , so Eq. (79a) is applicable. The constants  $K_1$  and  $K_2$  can be evaluated if the initial conditions  $q_0 = q_{sT}$  and

$i_0 = i_{E_T}$  at  $t = T$  are utilized. Inserting  $q_0 = q_{E_T}$  and  $t = T$  into Eq. (79a) yields one equation for  $K_7$  and  $K_8$ .

$$q_{E_T} = (K_7 - K_8 T) \epsilon^{-MT}$$

Another equation is obtained by differentiating Eq. (79a) and inserting the condition  $i_0 = i_{E_T}$  at  $t = T$ .

$$i_{E_T} = \left( \frac{dq_E}{dt} \right)_T = -M \epsilon^{-MT} \left( K_7 - K_8 T + \frac{K_8}{M} \right)$$

In these equations the values of  $q_{E_T}$  and  $i_{E_T}$  can be obtained from Eq. (81).

$$\begin{aligned} q_{E_T} &= CE + (K_3 - K_4 T) \epsilon^{-MT} \\ i_{E_T} &= -M \epsilon^{-MT} \left( K_3 - K_4 T + \frac{K_4}{M} \right) \end{aligned}$$

Solve for  $K_7$  and  $K_8$ .

$$\begin{aligned} K_7 &= CE(1 - MT) \epsilon^{MT} - CE = K_3[1 - (1 - MT) \epsilon^{MT}] \\ K_8 &= MCE(1 - \epsilon^{MT}) = K_4(1 - \epsilon^{MT}) \end{aligned}$$

Substitute these values into Eq. (89).

$$e_0 = E \epsilon^{-Mt} \{ \epsilon^{Mt} [1 + M(t - T)] - (1 + Mt) \} \quad (92)$$

This equation is the complete solution for output voltage in terms of the secondary parameters and pulse width for the critically damped case and after the generator pulse disappears.

When the pulse width is large compared with  $1/M$ , Eq. (92) can be simplified because  $\epsilon^{MT}$  is extremely large. Neglecting  $(1 + Mt)$ , Eq. (92) becomes

$$e_0 \approx E[1 + M(t - T)] \epsilon^{-M(t-T)} \quad (92a)$$

**4. Network Behavior;  $t = T$  to  $t = \infty$ .**—Equations (90), (91), and (92) are exceptionally cumbersome, and it is well to pause at this point to examine what the results mean in terms of the network in Fig. 68 for the three cases. The output-voltage equations for the condition that the generator pulse is large compared with  $1/M$  are simpler than the equations that apply for small pulse widths. Therefore, this discussion will be confined to the special case where the generator pulse is sufficiently large for Eqs. (90a), (91a), and (92a) to be applicable. Although generality is lost by doing this, the essential features

of the output-voltage behavior after the pulse disappears will be disclosed.

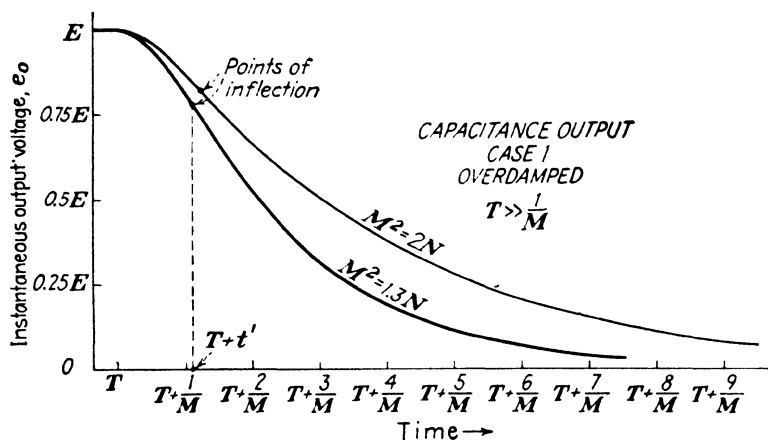


FIG. 74.—Output voltage of the network in Fig. 68 for the overdamped case and after the generator pulse disappears.

**Case 1. Overdamped.**—If the pulse width is large enough for the transient to be negligible at  $t = T$ , then Eq. (90a) applies.

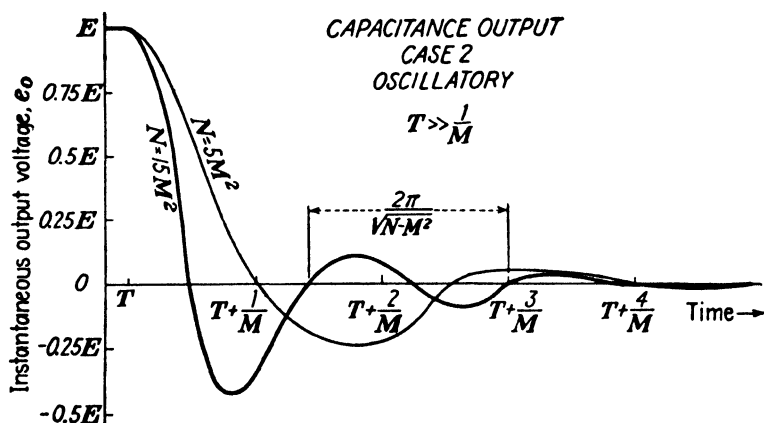


FIG. 75.—Output voltage of the network in Fig. 68 for the oscillatory case and after the generator pulse disappears.

Figure 74 is derived from Eq. (90a) for two values of  $M^2$  each of which is greater than  $N$ . The voltage is seen to decrease slowly from its initial value  $E$ , then to decrease more rapidly,

and finally to approach the steady-state value of zero. The charge on  $C$  decreases in a similar manner.

Inspection of Eq. (90a) bears out this behavior, and a comparison of Eq. (90a) with Eq. (84) shows a similarity that should not be overlooked. Equation (90a) is exactly equal to the negative of the transient term of Eq. (84) when  $t$  is replaced by  $(t - T)$ . In other words, the transient behavior of the network is the same on discharge as on charge of the capacitance. A similar comparison can be made in terms of the figures that represent the output voltage. If the curves in Fig. 74 are

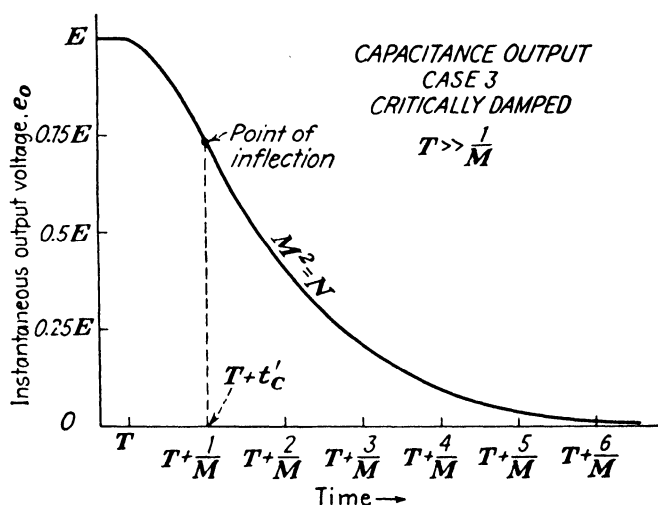


FIG. 76.—Output voltage of the network in Fig. 68 for the critically damped case and after the generator pulse disappears.

inverted so that the initial value at  $t = T$  is zero and the steady-state value is  $E$ , these inverted curves will be exactly the same as those shown in Fig. 69, but displaced in time by an amount  $T$ .

**Case 2. Oscillatory.**—Equation (91a) is presented graphically in Fig. 75 for the case of a generator-pulse width that is very large compared with  $1/M$ . The output voltage is oscillatory. The amplitude of the oscillation decreases exponentially and finally approaches the steady-state value of zero.

Equation (91a) lends itself readily to an interpretation of the oscillatory output because of the sinusoidal term. Moreover, it is clear from the equation that the amplitude of the

sinusoidal term diminishes exponentially. The period of oscillation is the same as that given by Eq. (87). As a matter of fact, Eq. (91a) is nothing more than the negative of the transient term of Eq. (85) with  $t$  replaced by the new variable  $(t - T)$ .

**Case 3. Critically Damped.**—Figure 76 represents Eq. (92a) for the case where the generator-pulse width has been long enough to allow the capacitor to charge up to the voltage  $E$ . Equation (92a) is simply the negative of the transient term of Eq. (86) that was illustrated in Fig. 73. Notice again that the steady-state value is attained more quickly in this case than in the overdamped case.

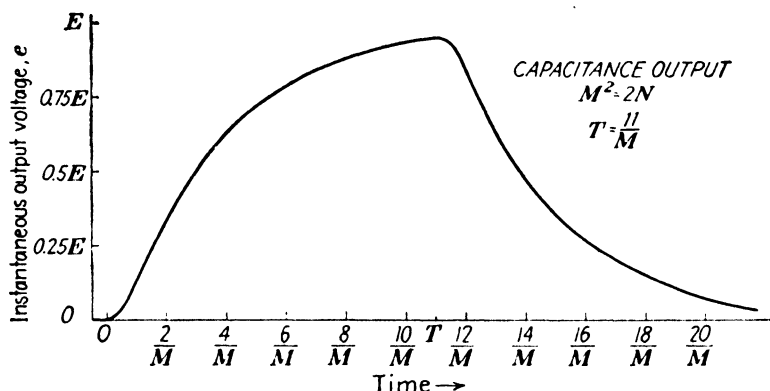


FIG. 77.—Output pulse of the network in Fig. 68 for the overdamped case.

**5. General Output Voltage;  $t = 0$  to  $t = \infty$ .**—To include output voltages for any pulse width and to consolidate the output voltage equations obtained, the complete pulse-response characteristic will be considered now. This can be accomplished by a graphical representation of the output pulse for various generator-pulse widths of constant amplitude.

Figure 77 indicates the output pulse that results in the overdamped case,  $M^2 = 2N$ , when the generator-pulse width is  $11/M$ . At  $t = T$  the output voltage has not yet reached the generator voltage even though the pulse width is eleven times  $1/M$ . The network behavior is even more sluggish for values of  $M^2$  that are greater than  $2N$ . The output pulse in the critically damped case is similar to that presented in Fig. 77, except that the voltage rises and falls a little more rapidly.

Figure 78 is an example of the output pulse for the oscillatory case when the generator-pulse width is equal to  $5/M$ . In

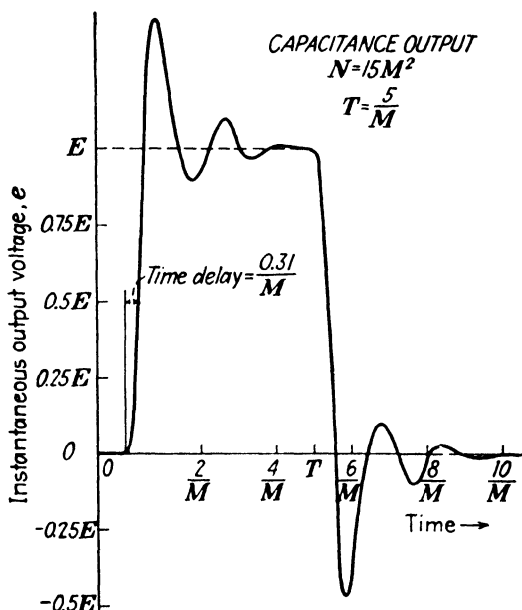


FIG. 78.—Output pulse of the network in Fig. 68 for the oscillatory case.

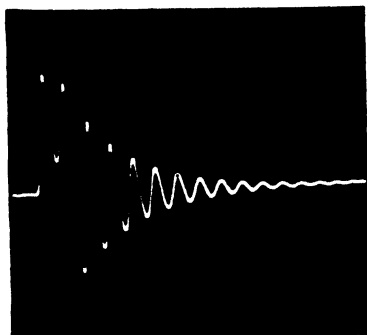


FIG. 79.—Oscillatory output pulse of the network in Fig. 68 when the transient is very large at  $t = T$  and for  $M^2 < N$ .



FIG. 80.—Oscillatory output pulse of the network in Fig. 68 when the transient is quite small at  $t = T$ .

this instance the transient is practically zero at  $t = T = 5/M$ . Observe that the output pulse is slightly displaced in time as compared with the generator pulse. When such a time delay

exists in a network, the time at which the pulse arrives at the output is frequently defined as the time when the output amplitude equals one-half the steady-state value. In Fig. 78 the output-pulse voltage is  $\frac{1}{2}E$  at approximately  $0.31/M$  sec.

Figures 79, 80, and 81 are illustrations of output pulses observed in actual *RLC* networks. In Fig. 79 the prolonged

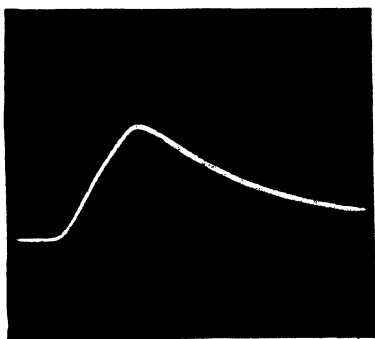


FIG. 81.—Output pulse of the network in Fig. 68 for the overdamped case when the generator pulse is  $2/M$ . This output pulse is approximately the integral of the generator pulse.

oscillation is due to the fact that  $M^2$  is much less than  $N$ . The generator-pulse width is slightly less than the interval between  $t = 0$  and the first time that the output voltage crosses zero voltage. The exponential nature of the output-voltage envelope is indicated clearly after the time  $t = T$ . Figure 80 pictures the output pulse that results in the oscillatory case when the transient is not quite negligible at  $t = T$ . Figure 81 depicts an output voltage in the overdamped case when the generator-pulse width is approximately

equal to  $2/M$ . This pulse suggests that the *RLC* network is capable of producing an output voltage that approaches the integral of the generator voltage.

#### BASIC *RLC* NETWORK WITH RESISTANCE ACROSS OUTPUT

If the network of Fig. 68 is rearranged as shown in Fig. 82, so that the output pulse appears across  $R$  instead of across  $C$ , the pulse-response characteristics will be different from those already obtained. The fundamental mathematical work on series *RLC* networks has really been carried out for the case of a capacitance output. Now it is possible to manipulate the preceding output-voltage equations to find the pulse-response characteristics for a resistance output. This mathematical manipulation is basically simple, but the expressions that arise are again cumbersome for this type of network.

**6. Equations for Output Pulse;  $t = 0$  to  $t = T$ .**—The output voltage during the time interval  $t = 0$  to  $t = T$  for the network

in Fig. 82 can be obtained if the equation for instantaneous current is known, since  $e_E = Ri_E$ . The instantaneous-current equation can be found without recourse to the differential equation for the network, because the equation for instantaneous voltage across  $C$  has been previously determined. When the equation for output voltage across  $C$  is multiplied by  $C$ , it becomes the equation for instantaneous charge. Differentiation of the instantaneous-charge equation converts it to an instantaneous-current equation. Finally, multiplication of the current equation by  $R$  yields the equation for voltage across  $R$ , which is the output voltage of the network in Fig. 82. In this manner, the three cases will be dealt with separately.

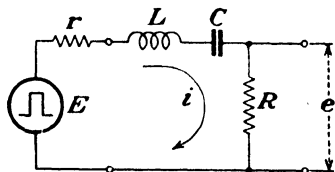


FIG. 82.—Series  $RLC$  network with a rectangular-pulse generator. Resistance output.

**Case 1.**  $M^2 > N$ . *Overdamped*.—Multiply Eq. (84) by  $C$  to obtain the equation for instantaneous charge.

$$q_E = Ce_E = CE - \frac{CE\epsilon^{-Mt}}{\sqrt{(M^2 - N)/N}} \sinh(\sqrt{M^2 - N}t + \alpha)$$

$$\alpha = \tanh^{-1} \frac{\sqrt{M^2 - N}}{M}$$

Differentiate to find the instantaneous-current equation.

$$i_E = \frac{CE\epsilon^{-Mt}}{\sqrt{(M^2 - N)/N}} [-\sqrt{M^2 - N} \cosh(\sqrt{M^2 - N}t + \alpha) + M \sinh(\sqrt{M^2 - N}t + \alpha)]$$

The following identity was proved in Chap. II:

$$P \sinh At - S \cosh At \equiv \sqrt{P^2 - S^2} \sinh \left( At - \tanh^{-1} \frac{S}{P} \right)$$

Therefore,

$$i_E = \frac{CE\epsilon^{-Mt}}{\sqrt{(M^2 - N)/N}} \left[ \sqrt{M^2 - M^2 + N} \sinh \left( \sqrt{M^2 - N}t + \alpha - \tanh^{-1} \frac{\sqrt{M^2 - N}}{M} \right) \right]$$

$$\begin{aligned}
 i_E &= \frac{NCE\epsilon^{-Mt}}{\sqrt{M^2 - N}} \sinh \sqrt{M^2 - N} t \\
 i_E &= \frac{2EM\epsilon^{-Mt}}{(R+r)\sqrt{M^2 - N}} \sinh \sqrt{M^2 - N} t \quad (93)
 \end{aligned}$$

The output voltage is  $Ri_E$ .

$$e_E = \frac{2ERM\epsilon^{-Mt}}{(R+r)\sqrt{M^2 - N}} \sinh \sqrt{M^2 - N} t \quad (94)$$

**Case 2.**  $M^2 < N$ . *Oscillatory*.—When Eq. (85) is multiplied by  $C$ , the equation for instantaneous charge results.

$$\begin{aligned}
 q_E = Ce_E = CE - \frac{CE\epsilon^{-Mt}}{\sqrt{(N - M^2)/N}} \sin (\sqrt{N - M^2} t + \beta) \\
 \beta = \tan^{-1} \frac{\sqrt{N - M^2}}{M}
 \end{aligned}$$

Differentiate.

$$\begin{aligned}
 i_E = \frac{dq_E}{dt} &= \frac{CE\epsilon^{-Mt}}{\sqrt{(N - M^2)/N}} \\
 [M \sin (\sqrt{N - M^2} t + \beta) - \sqrt{N - M^2} \cos (\sqrt{N - M^2} t + \beta)]
 \end{aligned}$$

Recall the following identity from Chap. II:

$$P \sin At - S \cos At \equiv \sqrt{P^2 + S^2} \sin \left( At - \tan^{-1} \frac{S}{P} \right)$$

Consequently,

$$\begin{aligned}
 i_E &= \frac{CE\epsilon^{-Mt}}{\sqrt{(N - M^2)/N}} \\
 \left[ \sqrt{M^2 + N - M^2} \sin \left( \sqrt{N - M^2} t + \beta - \tan^{-1} \frac{\sqrt{N - M^2}}{M} \right) \right] \\
 i_E &= \frac{2EM\epsilon^{-Mt}}{(R+r)\sqrt{N - M^2}} \sin \sqrt{N - M^2} t \quad (95)
 \end{aligned}$$

The output voltage is  $Ri_E$ .

$$e_E = \frac{2ERM\epsilon^{-Mt}}{(R+r)\sqrt{N - M^2}} \sin \sqrt{N - M^2} t \quad (96)$$

**Case 3.**  $M^2 = N$ . *Critically Damped*.—Multiplication of Eq. (86) by  $C$  yields the equation for instantaneous charge.

$$q_E = CE - CE(1 + Mt)\epsilon^{-Mt}$$

Differentiate.

$$\begin{aligned}
 i_E &= \frac{dq_E}{dt} = -CE[M - M(1 + Mt)]e^{-Mt} \\
 &= CEM^2te^{-Mt} = CENte^{-Mt} \\
 i_E &= \frac{E}{L} te^{-Mt} = \frac{2EM}{R+r} te^{-Mt}
 \end{aligned} \tag{97}$$

Multiply by  $R$ .

$$e_E = Ri_E = \frac{2ERMte^{-Mt}}{R+r} \tag{98}$$

**7. Output Voltage;  $t = 0$  to  $t = T$ .**—A graphical representation of the three output-voltage equations, combined with an

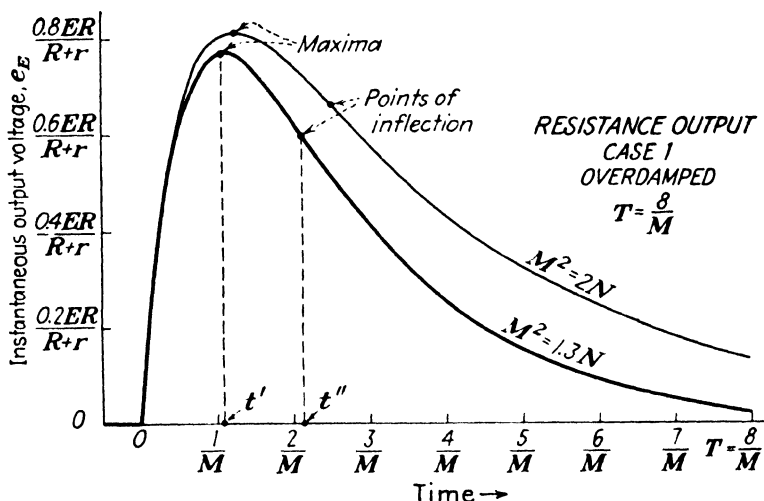


FIG. 83.—Output voltage of the network in Fig. 82 for the overdamped case and during the generator pulse.

interpretation of the equations themselves, discloses several pertinent features that lead to an understanding of the network behavior.

**Case 1. Overdamped.**—Figure 83 is derived from Eq. (94) for two values of  $M^2$ , each of which is greater than  $N$ . There is one striking property of this output voltage that is immediately apparent; namely, the output voltage is a pulse that rises from zero to a maximum and then falls to zero even though the generator voltage has a constant value. If one considers the slope of the curves of Fig. 69, however, this behavior is not

surprising. In fact, the point of inflection on the output-voltage curve of Fig. 69 corresponds to the instant that the current is a maximum. The fact that the current rises from zero to a maximum and then falls to zero can also be deduced by examining Eq. (93). At  $t = 0$  the current is zero because  $\sinh 0 = 0$ . The current is also zero after a long time has elapsed, since the limit of  $(\epsilon^{-Mt} \sinh \sqrt{M^2 - N} t)$  as  $t$  approaches  $\infty$  is zero when  $M^2 > N$ .<sup>1</sup> These two facts combined with the knowledge that charge accumulates on  $C$  mean that current must flow at some time between its two zero values. In other words, the current must pass through a maximum.

The maximum value of output voltage and the precise time at which the maximum occurs can be found from Eq. (94) by differentiating and setting  $de_E/dt$  equal to zero.

$$\begin{aligned} \frac{de_E}{dt} &= \frac{d}{dt} \left[ \frac{2ERM\epsilon^{-Mt}}{(R+r)\sqrt{M^2-N}} \sinh \sqrt{M^2-N} t \right] = 0 \\ \sqrt{M^2-N} \cosh \sqrt{M^2-N} t' - M \sinh \sqrt{M^2-N} t' &= 0 \\ \tanh \sqrt{M^2-N} t' &= \frac{\sqrt{M^2-N}}{M} = \tanh \alpha \end{aligned}$$

The time  $t'$  at which the output voltage is a maximum is

$$t' = \frac{\alpha}{\sqrt{M^2-N}} \quad (99)$$

The maximum value of output voltage can be found by substituting this value of  $t'$  into Eq. (94).

$$\begin{aligned} e_{E_{\max}} &= \frac{2ERM\epsilon^{-\frac{M\alpha}{\sqrt{M^2-N}}}}{(R+r)\sqrt{M^2-N}} \sinh \alpha \\ e_{E_{\max}} &= \frac{2ERM\epsilon^{-\frac{M\alpha}{\sqrt{M^2-N}}}}{(R+r)\sqrt{N}} \quad (100) \end{aligned}$$

It can now be seen that the generator-pulse width must be at least equal to  $t'$  in order to attain this maximum output voltage.

**Case 2. Oscillatory.**—Equation (96) is represented graphically in Fig. 84 for two values of  $M^2$  each of which is less than  $N$ , and for a generator-pulse width that is long enough to disclose the complete output-voltage variation during the generator

<sup>1</sup> See p. 126.

pulse. The output voltage oscillates about the steady-state value of zero with diminishing amplitude. Equation (96) can be regarded as consisting of the product of two terms: one, a

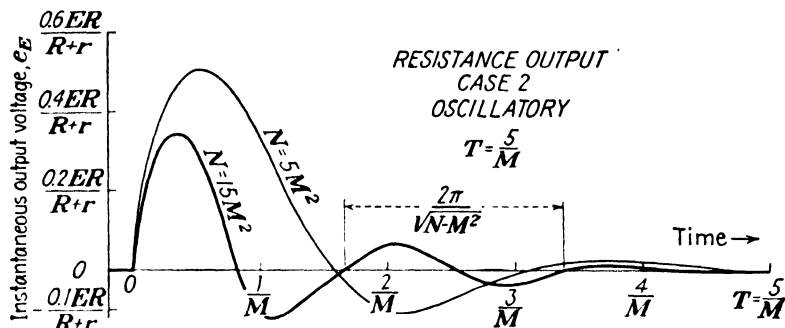


FIG. 84.—Output voltage of the network in Fig. 82 for the oscillatory case and during the generator pulse.

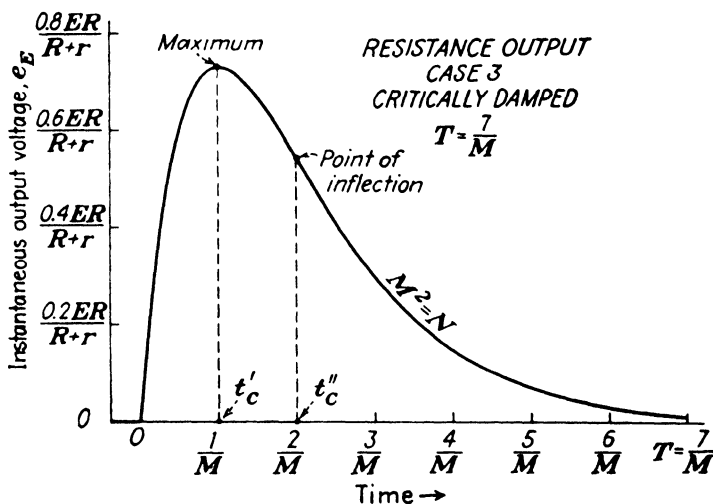


FIG. 85.—Output voltage of the network in Fig. 82 for the critically damped case and during the generator pulse.

sinusoidal term; the other, an exponential. The graphical product of a sine wave and an exponential was given in Fig. 71 and explains the output-voltage shape in Fig. 84.

The period of oscillation can be obtained from the angular velocity of the sine wave,  $\sqrt{N - M^2}$ . This angular velocity is the same as that in Eq. (85), so the period is given by Eq. (87).

As  $M^2$  becomes smaller compared with  $N$ , the output voltage oscillates with a smaller period. The period approaches infinity as the value of  $M^2$  approaches  $N$ . In the limit,  $M^2 = N$ , there is no oscillation and the critically damped case exists.

**Case 3. Critically Damped**—Equation (98) is shown in Fig. 85. The network is just on the verge of oscillation, but the output pulse still bears a close resemblance to that in the overdamped case. A close examination of Figs. 85 and 83 reveals that the output pulse in this case is sharper than in the overdamped case; in other words, the output pulse reaches a maximum more quickly and the steady-state value is attained sooner than in the overdamped case.

The time at which maximum output occurs can be found by setting the derivative of Eq. (98) equal to zero.

$$\begin{aligned}\frac{de_E}{dt} &= \frac{d}{dt} \left( \frac{2ERMt\epsilon^{-Mt}}{R+r} \right) = 0 \\ \frac{2ERM\epsilon^{-Mt'_c}}{R+r} (1 - Mt'_c) &= 0 \\ t'_c &= \frac{1}{M}\end{aligned}$$

It is informative to compare this time with that in the overdamped case:  $t' = \alpha/\sqrt{M^2 - N}$ . In general,  $t' > t'_c$ , which means that the critically damped output voltage reaches a maximum before the overdamped output voltage reaches a maximum. The difference between  $t'$  and  $t'_c$  becomes smaller as  $M^2$  approaches  $N$  in the overdamped case. This can be demonstrated by recalling that

$$\tanh \alpha = \frac{\sqrt{M^2 - N}}{M} = \sqrt{1 - \frac{N}{M^2}}$$

Now as  $M^2$  approaches  $N$ ,  $\tanh \alpha \approx \alpha$  because  $\sqrt{1 - N/M^2}$  becomes very small. Therefore,

$$t' = \frac{\alpha}{\sqrt{M^2 - N}} \approx \frac{\tanh \alpha}{\sqrt{M^2 - N}} = \frac{\sqrt{M^2 - N}}{M \sqrt{M^2 - N}} = \frac{1}{M} = t'_c$$

The maximum output voltage in the critically damped case can be found by substituting  $t = t'_c = 1/M$  into Eq. (98).

$$e_{E_{\max}} = \frac{2ERM\epsilon^{-1}}{(R+r)M} = 0.736 \frac{ER}{(R+r)}$$

**8. Equations for Output Pulse ;  $t = T$  to  $t = \infty$ .**—The output-voltage equations that apply after the generator pulse has disappeared can be obtained by the same method employed in the time interval  $t = 0$  to  $t = T$ . Multiply Eqs. (90), (91), and (92) by  $C$ ; differentiate each equation, and multiply by  $R$ . The resulting equations are

$$\text{Case 1. } e_0 = - \frac{2ERM\epsilon^{-Mt}}{(R+r)\sqrt{M^2-N}} \\ [\epsilon^{Mt} \sinh \sqrt{M^2-N}(t-T) - \sinh \sqrt{M^2-N}t] \quad (101)$$

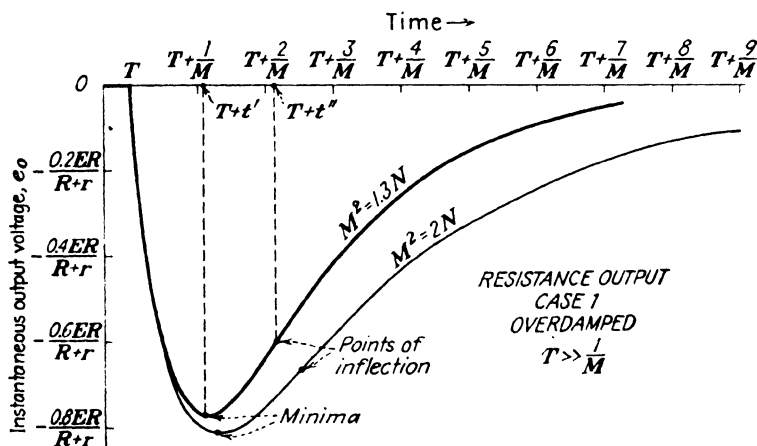


FIG. 86.—Output voltage of the network in Fig. 82 for the overdamped case and after the generator pulse disappears when  $T \gg 1/M$ .

$$\text{Case 2. } e_0 = - \frac{2ERM\epsilon^{-Mt}}{(R+r)\sqrt{N-M^2}} \\ [\epsilon^{Mt} \sin \sqrt{N-M^2}(t-T) - \sin \sqrt{N-M^2}t] \quad (102)$$

$$\text{Case 3. } e_0 = \frac{2ERM\epsilon^{-Mt}}{R+r} [(T-t)\epsilon^{Mt} + Mt] \quad (103)$$

These equations simplify if the pulse width is very large compared with  $1/M$ .

$$\text{Case 1. } e_0 \approx \frac{-2ERM\epsilon^{-M(t-T)}}{(R+r)\sqrt{M^2-N}} \sinh \sqrt{M^2-N}(t-T) \quad (101a)$$

$$\text{Case 2. } e_0 \approx \frac{-2ERM\epsilon^{-M(t-T)}}{(R+r)\sqrt{N-M^2}} \sin \sqrt{N-M^2}(t-T) \quad (102a)$$

**Case 3.**  $e_0 \approx -\frac{2ERM(t-T)\epsilon^{-M(t-T)}}{R+r}$  (103a)

**9. Output Voltage;  $t = T$  to  $t = \infty$ .**—Figures 86, 87, and 88 show the output voltage after the time  $t = T$  for the three cases,

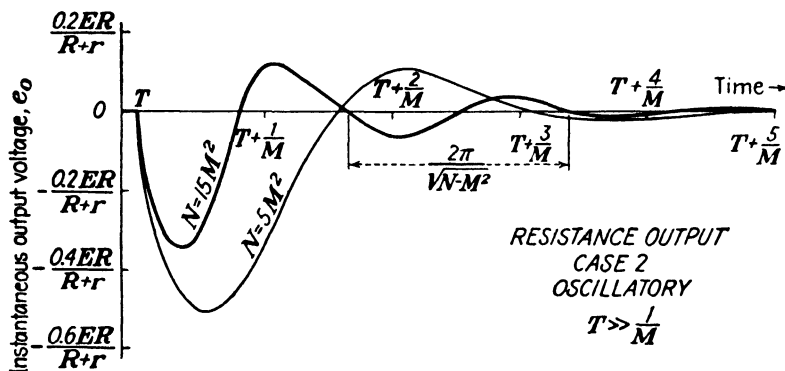


FIG. 87.—Output voltage of the network in Fig. 82 for the oscillatory case and after the generator pulse disappears when  $T \gg 1/M$ .

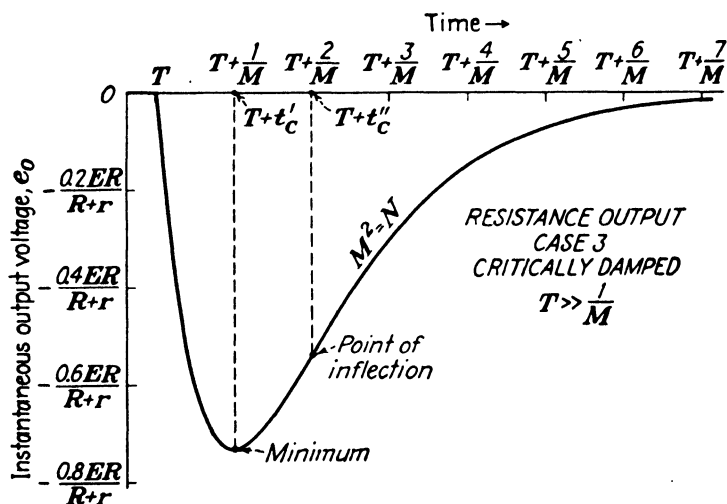


FIG. 88.—Output voltage of the network in Fig. 82 for the critically damped case and after the generator pulse disappears when  $T \gg 1/M$ .

assuming that the generator pulse has been long enough for Eqs. (101a), (102a), and (103a) to apply. A comparison of these output voltages with those of Figs. 83, 84, and 85 reveals that they are exactly the same except they are inverted and shifted

in time by an amount  $T$ . In other words, the network behavior is the same, but the direction of the current has been reversed.

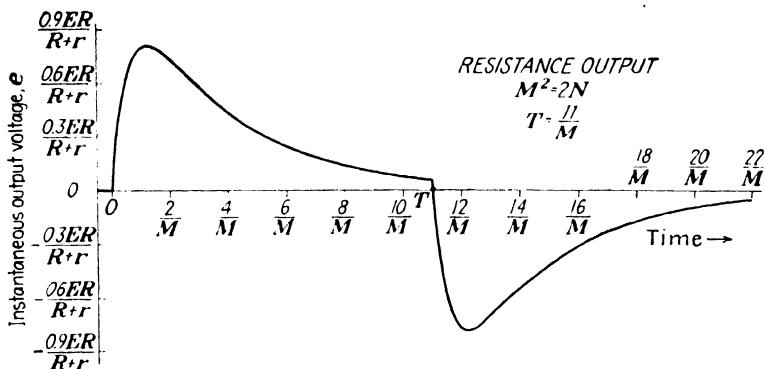


FIG. 89.—Output pulse of the network in Fig. 82 for the overdamped case.

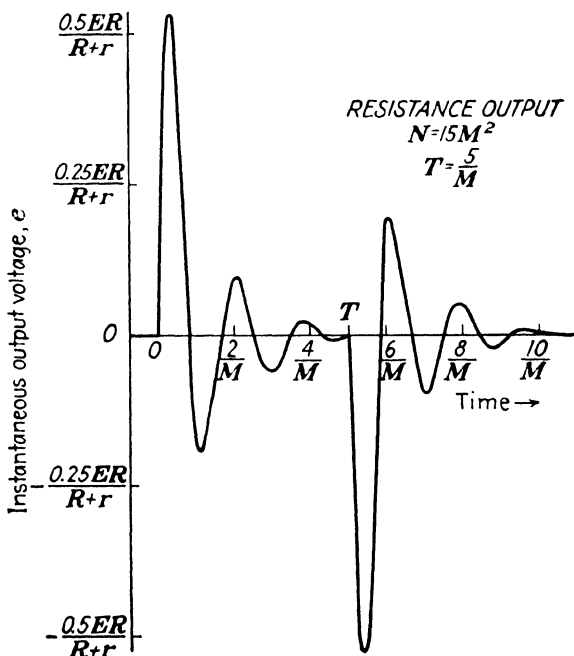


FIG. 90.—Output pulse of the network in Fig. 82 for the oscillatory case.

A comparison of Eqs. (101a), (102a), and (103a) with Eqs. (94), (96), and (98) discloses that the only difference is that of sign and the replacement of the variable  $t$  by  $(t - T)$ .

**10. General Output Voltage;  $t = 0$  to  $t = \infty$ .**—The output voltages for the two time intervals can be combined to represent

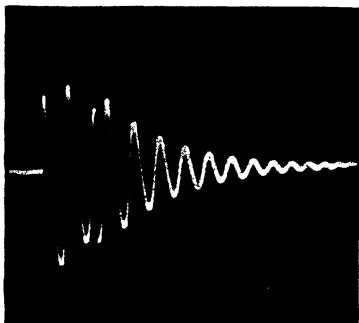


FIG. 91.—Oscillatory output voltage of the network in Fig. 82 when the transient is very large at  $t = T$  and  $M^2 \ll N$ .

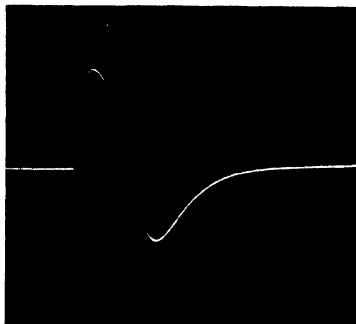


FIG. 92.—Output pulse of the network in Fig. 82 for the overdamped case when the transient is large at  $t = T$ .

the complete output pulse. This has been done for two cases in Figs. 89 and 90.

Figure 89 represents the output pulse that results in the overdamped case ( $M^2 = 2N$ ) for a generator pulse of amplitude  $E$  and duration  $T = 11/M$ . The maximum output voltage is only about  $0.81E$ . At  $t = 11/M$  the transient term is still significant and the output voltage is approximately  $0.06E$ . The output pulse for the critically damped case is similar in shape, but the maximum output voltage is smaller and the transient diminishes more rapidly than in the overdamped case.

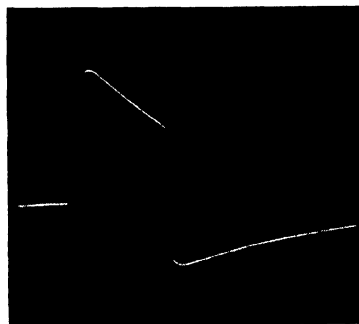


FIG. 93.—Output pulse of the network in Fig. 82 for the overdamped case when  $M^2$  is extremely large compared with  $N$ .

The output pulse for the oscillatory case ( $N = 15M^2$ ) and for a generator pulse of amplitude  $E$  and duration  $T = 5/M$  is given in Fig. 90. There is practically no resemblance between the output pulse and the generator pulse. The transient has diminished to a fairly small value at  $t = T$ .

Figures 91, 92, and 93 are oscillograms that provide additional examples of the pulse voltage that can appear across  $R$  in an  $RLC$  series network. In Fig. 91 the abrupt change in oscillatory output voltage at  $t = T$  is the result of the sudden disappearance of the generator pulse. The transient endures for a considerable length of time after the generator pulse has become zero. Figures 92 and 93 are illustrations of output pulses that can result in the overdamped case when the transient is very large at  $t = T$ .

#### BASIC $RLC$ NETWORK WITH INDUCTANCE ACROSS OUTPUT

A third possible arrangement of the network in Fig. 68 is that in Fig. 94 where the output pulse appears across the inductance. The instantaneous-current equations that have been previously determined can be manipulated into the desired equations for voltage across  $L$  since the network current is the same in both cases.

**11. Equations for Output Pulse ;**  
 $t = 0$  to  $t = T$ .—When Eqs. (93), (95), and (97) are differentiated and multiplied by  $L$ , the output-voltage equations for the network shown in Fig. 94 are obtained.

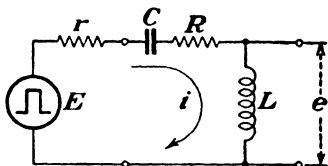


FIG. 94.—Series  $RLC$  network with a rectangular-pulse generator. Inductance output.

The differentiated current equations are simplified in a manner similar to that employed on page 139, and the following output-voltage equations result:

$$\text{Case 1. } e_E = - \frac{E\epsilon^{-Mt}}{\sqrt{(M^2 - N)/N}} \sinh(\sqrt{M^2 - N} t - \alpha) \quad (104)$$

$$\text{Case 2. } e_E = - \frac{E\epsilon^{-Mt}}{\sqrt{(N - M^2)/N}} \sin(\sqrt{N - M^2} t - \beta) \quad (105)$$

$$\text{Case 3. } e_E = E(1 - Mt)\epsilon^{-Mt} \quad (106)$$

These equations apply only during the time interval  $t = 0$  to  $t = T$  because Eqs. (93), (95), and (97) are valid only during this time interval.

**12. Output Voltage ;  $t = 0$  to  $t = T$ .**—An understanding of the output voltage equations can be obtained by a graphical representation of the equations and by comparing them with those previously derived.

**Case 1. Overdamped.**—Figure 95 is a graphical representation of Eq. (104) for two different values of  $M^2$  each of which is greater than  $N$ . The output voltage is different from any obtained previously. At  $t = 0$  all of the generator voltage  $E$  appears across  $L$ , but as time increases, this voltage decreases to zero and reverses polarity (becomes negative). After reaching a maximum negative value, the voltage subsequently diminishes to the steady-state value of zero. To explain this output-voltage behavior in graphical terms, recall that the voltage

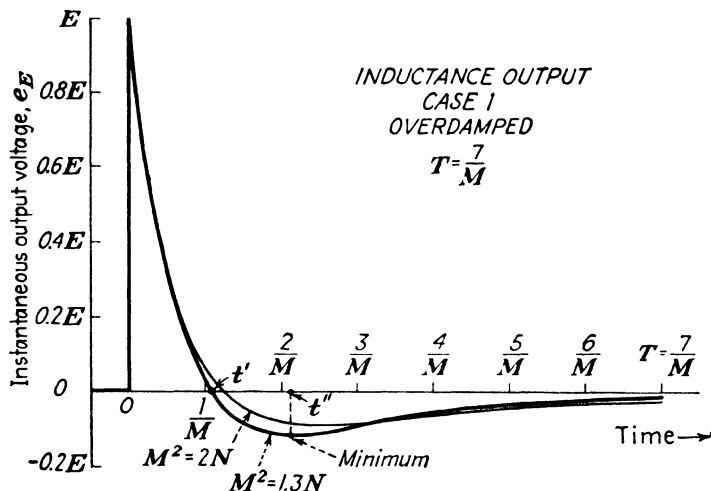


FIG. 95.—Output voltage of the network in Fig. 94 for the overdamped case and during the generator pulse.

across  $L$  is proportional to the rate of change of instantaneous current. The curves in Fig. 83 are directly proportional to the current, and their slope is proportional to the voltage across  $L$ .

The time at which the output voltage crosses zero corresponds to the instant at which the current in the network is a maximum, since the output voltage in this case is proportional to the derivative of the current. This time has been seen to be

$$t' = \frac{\alpha}{\sqrt{M^2 - N}}$$

The time  $t''$  at which the voltage is a maximum negative value corresponds to the point of inflection in the output-voltage curve

shown in Fig. 83 and can be found by setting the derivative of Eq. (104) equal to zero.

$$\begin{aligned} \frac{de_E}{dt} &= \frac{d}{dt} \left[ -\frac{E\epsilon^{-Mt}}{\sqrt{(M^2 - N)}/N} \sinh(\sqrt{M^2 - N}t - \alpha) \right] = 0 \\ \sqrt{M^2 - N} \cosh(\sqrt{M^2 - N}t'' - \alpha) &= 0 \\ -M \sinh(\sqrt{M^2 - N}t'' - \alpha) &= 0 \\ \tanh(\sqrt{M^2 - N}t'' - \alpha) &= \frac{\sqrt{M^2 - N}}{M} = \tanh \alpha \end{aligned}$$

Therefore,  $\sqrt{M^2 - N}t'' - \alpha = \alpha$

$$\text{or } t'' = \frac{2\alpha}{\sqrt{M^2 - N}} = 2t'$$

This implies that the time interval between  $t = 0$  and the time that the output voltage crosses zero is equal to the time interval between the crossover and the time of maximum negative output voltage. The value of the maximum negative output voltage can be found by substituting  $t''$  into Eq. (104).

$$\begin{aligned} (e_E)_{t''} &= -\frac{E\epsilon^{-Mt''}}{\sqrt{(M^2 - N)}/N} \sinh(\sqrt{M^2 - N}t'' - \alpha) \\ &= -\frac{E\epsilon^{-\frac{2M\alpha}{\sqrt{M^2 - N}}}}{\sqrt{(M^2 - N)}/N} \sinh(2\alpha - \alpha) \\ &= -E\epsilon^{-\frac{2M\alpha}{\sqrt{M^2 - N}}} \end{aligned}$$

A comparison of Eq. (104) with Eq. (84) discloses that the transient term of Eq. (84) is the same as Eq. (104) with the exception of the sign of  $\alpha$ .

The negative sign in front of Eq. (104) can be rather deceptive unless the complete equation is examined as a whole. For very small values of  $t$ , the hyperbolic sine term is negative and hence the output voltage is positive. However, after a time equal to  $t'$  has elapsed, the hyperbolic sine term is positive and consequently the output voltage is negative. Figure 95 shows this clearly.

**Case 2. Oscillatory.**—Equation (105) applies in this case, and it is shown in Fig. 96 for two values of  $M^2$  that are less than  $N$ . This is a damped oscillation, the damping being exponential and taking place in accordance with a time constant equal to

$1/M$ . The oscillation takes place about the steady-state value of zero output voltage. The period of oscillation is given by Eq. (87). The phase of the sinusoidal term is different from the

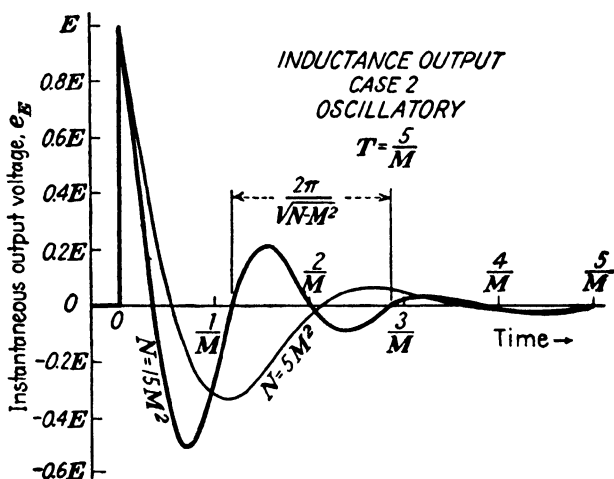


FIG. 96.—Output voltage of the network in Fig. 94 for the oscillatory case and during the generator pulse.

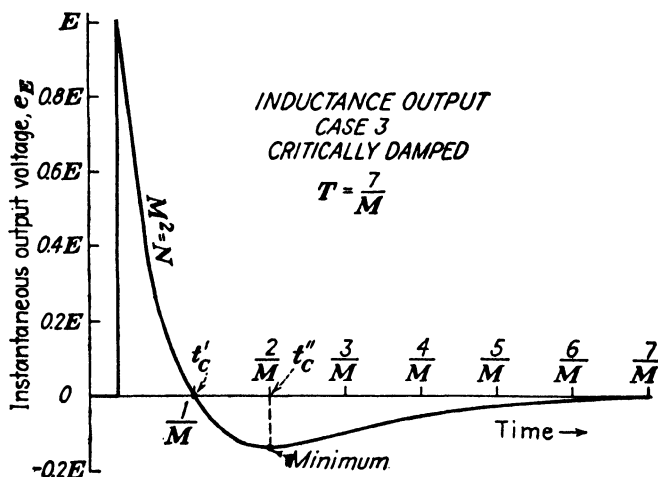


FIG. 97.—Output voltage of the network in Fig. 94 for the critically damped case and during the generator pulse.

oscillatory cases of capacitance and resistance outputs, as can be seen from Eqs. (96) and (85). Notice that the transient term of Eq. (85) is very similar to Eq. (105).

**Case 3. Critically Damped.**—Figure 97 is a graphical representation of Eq. (106). The output voltage is similar to the overdamped case even though the network is just on the verge of oscillation.

By inspection of Eq. (106), the voltage crossover point is seen to occur at a time when  $1 = Mt$  or when  $t = t'_c = 1/M$ . At this instant the current is a maximum as was deduced previously. Since this time is less than the crossover point in the

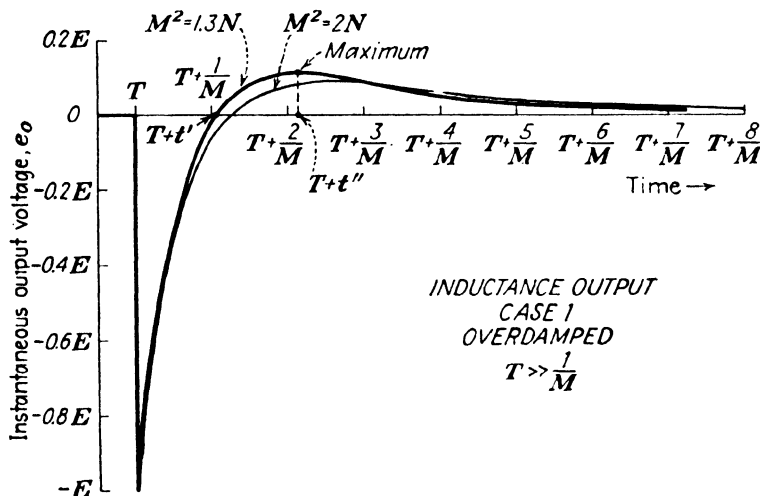


FIG. 98.—Output voltage of the network in Fig. 94 for the overdamped case and after the generator pulse disappears when  $T \gg 1/M$ .

overdamped case, as can be seen by comparing  $\alpha/\sqrt{M^2 - N}$  with  $1/M$  (page 144), it is evident that the transient diminishes more rapidly in the critically damped case than in the overdamped case.

The time at which the output voltage attains its most negative value can be found by setting the derivative of Eq. (106) equal to zero.

$$\begin{aligned} \frac{de_o}{dt} &= \frac{d}{dt} [E(1 - Mt)\epsilon^{-Mt}] = 0 \\ E\epsilon^{-Mt''c} [-M - M(1 - Mt''c)] &= 0 \\ t''c &= \frac{2}{M} = 2t'_c \end{aligned}$$

The ratio of the crossover time and the time of maximum negative value is the same as in the overdamped case.

$$\frac{t_c''}{t_c'} = \frac{t''}{t'} = 2$$

However,  $t_c'' < t''$  and  $t_c' < t'$ .

Since  $t_c'' = 2t_c'$ , then  $t_c'' - t_c' = t_c'$ . This shows that the time interval between  $t = 0$  and the time that the output voltage crosses zero is equal to the time interval between the crossover and the time of maximum negative output voltage.

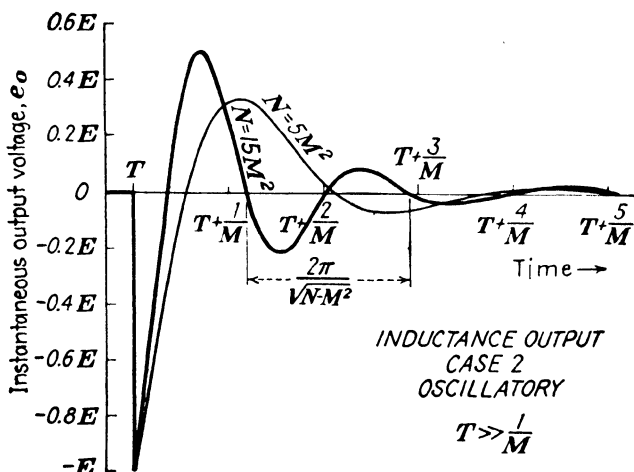


FIG. 99.—Output voltage of the network in Fig. 94 for the oscillatory case and after the generator pulse disappears when  $T >> 1/M$ .

**13. Equations for Output Pulse;  $t = T$  to  $t = \infty$ .**—To complete the solution for the network in Fig. 94, the output-voltage behavior must be determined from the time  $T$  on. Again this can be accomplished expediently by differentiation of the instantaneous-current equations and multiplication by  $L$ . The instantaneous-current equations can be obtained by dividing Eqs. (101), (102), and (103) by  $R$ . The resulting equations for output voltage are

$$\text{Case 1. } e_0 = \frac{Ee^{-Mt}}{\sqrt{(M^2 - N)/N}} \{ e^{Mt} \sinh [\sqrt{M^2 - N} (t - T) - \alpha] - \sinh (\sqrt{M^2 - N} t - \alpha) \} \quad (107)$$

$$\text{Case 2. } e_0 = \frac{E\epsilon^{-Mt}}{\sqrt{(N - M^2)/N}}$$

$$\{\epsilon^{MT} \sin [\sqrt{N - M^2}(t - T) - \beta] - \sin (\sqrt{N - M^2}t - \beta)\} \quad (108)$$

$$\text{Case 3. } e_0 = E\epsilon^{-Mt}[\epsilon^{MT}(Mt - 1) - MT - M^2t] \quad (109)$$

Since these equations are so cumbersome, it is best to discuss them in their reduced form for the case where the pulse width has been long enough for the transient term to be negligible at the time  $t = T$ . When  $T \gg 1/M$ , then  $\epsilon^{MT}$  is very large and the output-voltage equations become approximately

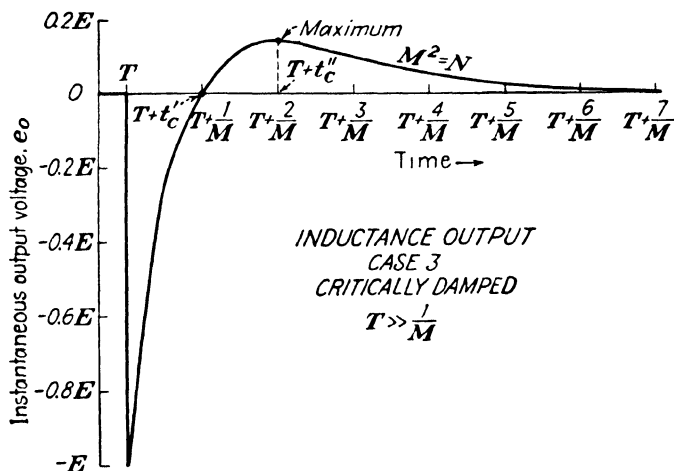


FIG. 100.-Output voltage of the network in Fig. 94 for the critically damped case and after the generator pulse disappears when  $T \gg 1/M$ .

$$\text{Case 1. } e_0 \approx \frac{E\epsilon^{-M(t-T)}}{\sqrt{(M^2 - N)/N}} \sinh [\sqrt{M^2 - N}(t - T) - \alpha] \quad (107a)$$

$$\text{Case 2. } e_0 \approx \frac{E\epsilon^{-M(t-T)}}{\sqrt{(N - M^2)/N}} \sin [\sqrt{N - M^2}(t - T) - \beta] \quad (108a)$$

$$\text{Case 3. } e_0 \approx E[M(t - T) - 1]\epsilon^{-M(t-T)} \quad (109a)$$

In this form, the striking similarity between these equations and the corresponding equations for the resistance and capacitance output is evident. See Eqs. (101a), (102a), and (103a), and especially Eqs. (90a), (91a), and (92a).

**14. Output Voltage;  $t = T$  to  $t = \infty$ .**—Equations (107a) (108a), and (109a) are shown in Figs. 98, 99, and 100, respectively. The output voltage is seen to be essentially the negative of the output voltage during the generator pulse but shifted in time by an amount  $T$ . Compare these output voltages with those given in Figs. 95, 96, and 97.

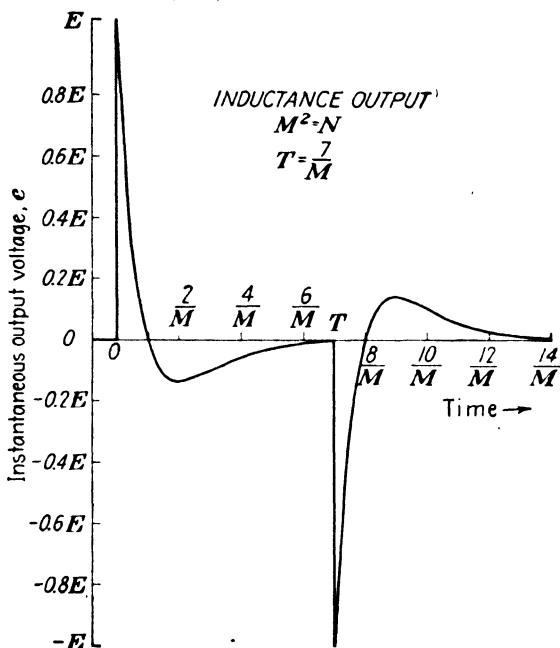


FIG. 101.—Output pulse of the network in Fig. 94 for the critically damped case.

**15. General Output Voltage;  $t = 0$  to  $t = \infty$ .**—The complete pulse-response characteristics of the network in Fig. 94 should now be clear. Figures 101 and 102 present two possible output voltages both during and after the generator pulse.

Figure 101 indicates the output voltage that results from a generator-pulse voltage  $E$  and width  $T = 7/M$  when the network is critically damped ( $M^2 = N$ ). The transient is negligible at  $t = 7/M$ . In the overdamped case the output voltage is similar in shape, but a longer generator-pulse width is required before the transient becomes negligible at  $t = T$ .

Figure 102 demonstrates an oscillatory output voltage when  $N = 15M^2$  and when the generator-pulse width is  $5/M$ . The

transient has time to diminish to a very small value at  $t = T$  for this generator-pulse width.

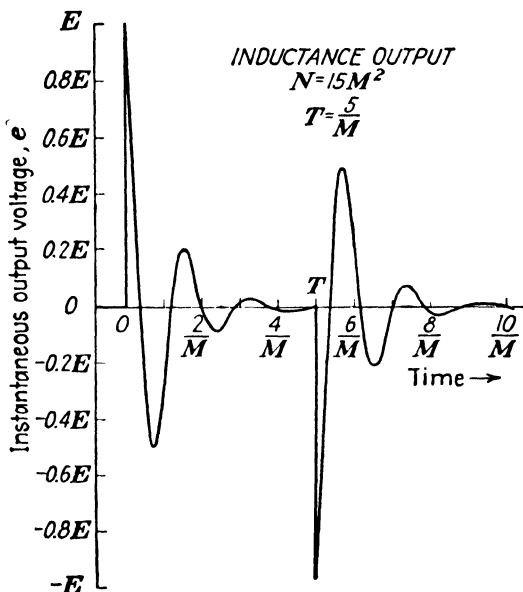


FIG. 102. —Output pulse of the network in Fig. 94 for the oscillatory case.

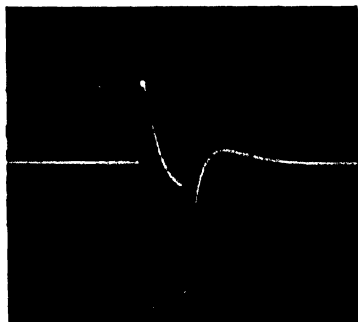


FIG. 103.—Output pulse of the network in Fig. 94 for the overdamped case when  $T = t''$ .

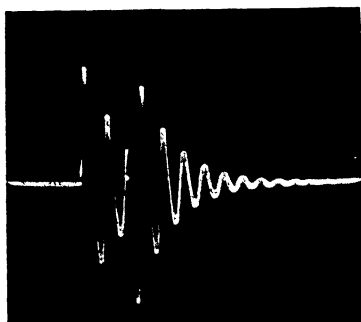


FIG. 104.—Oscillatory output voltage of the network in Fig. 94 when the transient is very large at  $t = T$  and  $M^2 \ll N$ .

Figure 103 is an oscillogram of the output voltage in the overdamped case. The generator-pulse width is adjusted for the condition  $T = t''$ ; in other words, the generator pulse becomes

zero at the instant the output voltage is a minimum during the generator pulse. The equation  $t'' = 2t'$  can be verified from Fig. 103 if linear measurements along the time axis are made.

The oscillogram in Fig. 104 represents the output voltage in the oscillatory case when the transient is still large at  $t = T$ .

### DESCRIPTIVE NETWORK BEHAVIOR

So far, the description of the network behavior has been largely mathematical, and only a small amount of attention has been paid to the behavior of current, charge, and energy in the network for the various cases. The graphical representation of the output voltage is always a helpful consideration, but a more penetrating inquiry into the network behavior requires that the physical situation be appraised. This can be done by using the graphical and mathematical results as a guide but not as a self-sufficient explanation of the network operation. A general discussion of the network in qualitative terms should serve also as a review of this chapter.

A description of the network logically divides into two time intervals: during the generator pulse, and after the generator pulse. Accordingly, the two states of operation will be discussed separately.

As far as energy relations are concerned, a mathematical analysis is possible, but a sufficient amount of information concerning the energy transfer and transformation can be obtained from qualitative considerations that utilize only the basic energy definitions. Consequently, a qualitative point of view will be taken as regards energy.

**16. During Generator Pulse.**—The first time interval to be considered is from  $t = 0$  to  $t = T$ . During this interval the generator pulse exists, and the voltage across  $C$  approaches a steady-state value that is determined by the generator voltage.

The initial conditions are  $q_E = 0$  and  $i_E = 0$  at  $t = 0$ . This means that at  $t = 0$  all of the generator voltage must appear across the inductance, because the voltage across  $(R + r)$  and across  $C$  is zero. These initial conditions apply irrespective of the relative values of  $M^2$  and  $N$ . However, the network behavior after  $t = 0$  will depend upon the network parameters.

**Case 1. Overdamped.**—As time increases, current flows and charge begins to accumulate on  $C$ . This means that voltage

begins to appear across  $(R + r)$  and across  $C$ ; hence the voltage across  $L$  must begin to decrease. The steady-state value of current is zero, and if the pulse width is sufficiently large, the current goes through a maximum. At the instant the current is a maximum, the voltage across  $L$  is zero because the rate of change of current is zero. Moreover, at this instant the rate of increase of charge on  $C$  is a maximum and the voltage across  $(R + r)$  is a maximum. As soon as the current begins to decrease, a voltage is developed across  $L$  that opposes the change in current. This voltage will be opposite in polarity to the voltage that existed across  $L$  during the time that the current was increasing. As the current diminishes (always flowing in the same direction), the voltage across  $(R + r)$  decreases while the voltage across  $C$  approaches the steady-state value  $E$ . At the instant the current is decreasing at a maximum rate, the voltage across  $L$  will be a maximum. In the steady-state, the voltage across  $L$  and across  $(R + r)$  is zero because the flow of current has ceased, and the voltage across  $C$  is equal to the generator voltage.

Energy considerations are informative, and the network behavior will now be discussed in terms of energy transfer and transformation. During the generator pulse, the generator delivers energy to the network. Eventually one-half of this energy is stored in the electric field of  $C$  and the remainder is dissipated as heat in  $(R + r)$  if the pulse width is very long. But the situation is not quite this simple. When current begins to flow, energy is dissipated in  $(R + r)$ , is stored in the magnetic field of  $L$ , and is stored in the electric field of  $C$ . When the current is a maximum, the instantaneous dissipation in  $(R + r)$  is a maximum and the energy stored in  $L$  is a maximum. The energy stored in  $L$  is subsequently released and transferred to the electric field of  $C$ , during which time additional energy is dissipated in  $(R + r)$ .

The value of  $(R + r)$  relative to  $2\sqrt{L/C}$  (or  $M^2$  relative to  $N$ ) is a critical factor. In this case  $(R + r)$  is greater than  $2\sqrt{L/C}$  and limits the flow of current and thereby limits the amount of energy that can be stored in the magnetic field of  $L$ . When this stored energy is released and combined with the energy already existing in the electric field of  $C$ , the total energy is always less than the steady-state energy corresponding to a

voltage  $E$  across  $C$ . Therefore,  $C$  never has energy in excess of the steady-state value and stores all that is delivered to it. When  $(R + r)$  is made smaller, as in Case 2, the energy situation is altered because it is possible for  $C$  to have an excess of energy that must be released.

If  $(R + r)$  is thought of as a fixed value, then making  $L$  small enough or  $C$  large enough to have  $(R + r) > 2\sqrt{L/C}$  indicates the same energy behavior. If  $L$  is made smaller, then for a given current the stored electromagnetic energy is less, and the energy released from  $L$  to  $C$  is less. If  $C$  is made larger, then the amount of energy that  $C$  is capable of storing is greater and the steady-state energy is greater. These possibilities lead to the qualitative conclusion that if  $(R + r) > 2\sqrt{L/C}$ , then the electrostatic energy stored by  $C$  is never in excess of its steady-state energy. It is important to appreciate that the actual values of  $(R + r)$ ,  $L$ , and  $C$  are insignificant, but that the *relative* value of  $(R + r)$  compared with  $2\sqrt{L/C}$  is the critical factor.

**Case 2. Oscillatory.**—In the oscillatory case the current increases from zero to a maximum value, then decreases to zero, reverses its direction, and reaches another maximum, and so on. The charge on  $C$  oscillates in a similar manner. Consequently, the voltage across  $(R + r)$ ,  $L$ , and  $C$  is oscillatory.

The energy relations are particularly useful to describe the network behavior. When the current reaches its first maximum, the energy stored in the magnetic field of  $L$  is a maximum. Some of this stored energy is released to  $C$  and some is dissipated in  $(R + r)$ . The addition of this released energy to the energy already stored in the electric field of  $C$  is sufficient to cause the total electrostatic energy to exceed the steady-state value. Therefore, part of the energy stored in  $C$  must be returned to the network, some being dissipated in  $(R + r)$  and the remainder being re-stored in the magnetic field of  $L$ . This energy exchange between  $L$  and  $C$  continues as does the corresponding transformation of energy from electromagnetic to electrostatic, but each time the transfer occurs part of the energy is dissipated in  $(R + r)$ . This is why the amplitude of the oscillation diminishes. Eventually, the energy in  $C$  settles at the steady-state value, and the electromagnetic energy becomes zero.

The condition for an oscillatory output is  $(R + r) < 2\sqrt{L/C}$ .

This condition can be satisfied for: (a) a fixed value of  $L$  and  $C$  with  $(R + r)$  sufficiently small, (b) a fixed value of  $L$  and  $(R + r)$  with  $C$  sufficiently small, (c) a fixed value of  $C$  and  $(R + r)$  with  $L$  sufficiently large. From an energy viewpoint the requirement for oscillation can be stated as the condition under which the initial energy release from  $L$  to  $C$  is sufficient to impart to  $C$  an energy greater than its steady-state value. With this criterion, the oscillation can be descriptively explained for the three possibilities above.

- a. If  $(R + r)$  is made small, then the current flow is large and the energy storage in  $L$  is large.
- b. If  $C$  is made small, then the steady-state value of energy is smaller and less electrostatic energy is required to cause the total electrostatic energy to be in excess of the steady-state value.
- c. If  $L$  is made large, it is capable of storing a large amount of energy.

For all possibilities, the condition for an amount of electrostatic energy greater than the steady-state value is encouraged.

**Case 3. Critically Damped.**—The behavior of the network in this case is very similar to the overdamped case. Here, however, the network is just on the verge of oscillation but is not actually oscillatory. This means that the energy that is imparted to  $C$  from  $L$  is just enough to bring the total electrostatic energy up to the steady-state value. In other words, the energy imparted to  $C$  from  $L$  is the maximum amount that will not result in oscillation. From this consideration it can be deduced that steady-state conditions are attained more quickly in the critically damped case than in the overdamped case.

**17. After Generator Pulse.**—The network behavior after the generator pulse disappears depends upon the conditions at  $t = T$ , which in turn depend upon the generator-pulse width and the network parameters.

If the pulse width is so large that the transient is negligible just before the pulse disappears, then the voltage across  $C$  is equal to the generator voltage and the voltage across  $(R + r)$  and  $L$  is zero. All the energy is contained in the electric field of  $C$ , and no magnetic field exists. When the generator pulse disappears, the energy stored in  $C$  is released to the network.

Some of this energy is dissipated in  $(R + r)$ , and some is stored in the magnetic field of  $L$ . The behavior of the energy depends upon the relative values of  $(R + r)$  and  $2\sqrt{L/C}$ . Eventually, all the energy stored in  $C$  is dissipated in  $(R + r)$  and the energy in  $L$  becomes zero.

If the pulse disappears from the generator before steady-state conditions are reached, the network behavior becomes more complicated. In general; both  $L$  and  $C$  will contain stored energy at  $t = T$ . In any case, however, the total energy in the network at  $t = T$  will be dissipated in  $(R + r)$  and all of the stored energy will become zero. In general, the energy supplied by the generator to the network is always eventually equal to the total energy dissipated in  $(R + r)$ .

### CONCLUSION

Essentially one series  $RLC$  network has been analyzed in this chapter, although the output voltages were taken across three different elements of the network. It is possible to analyze other series  $RLC$  networks where the output voltage is taken across two or three different parameters. However, such networks are not often encountered. Moreover, the equations for output voltage are much more cumbersome than those already derived. Therefore, it is not worth while to obtain the output-voltage equations for a perfectly general series  $RLC$  network.

It should be obvious that series  $RLC$  networks are more complex than series  $RL$  or series  $RC$  networks. In most cases networks that contain  $R$ ,  $L$ , and  $C$  are more complex than those that do not contain both inductance and capacitance. This will become especially evident in Chap. VIII. However, the additional complexity should not obscure the fact that only a small number of fundamental principles is required to effect a solution.

The dependence of the output-voltage shape upon the relative values of network parameters should reinforce a fundamental point; namely, when a sudden change is imposed upon a network, the nature of the transient is governed by the network parameters and not by the disturbing force.

This chapter concludes the treatment of series networks, but the analyses of all subsequent networks rely upon the results

of Chaps. III, IV, and V. It will be shown that many series-parallel networks behave exactly the same as series networks as far as their pulse-response characteristics are concerned. For this reason it is imperative that series networks be understood thoroughly before proceeding to series-parallel networks.

### Problems

**Prob. 1.** The parameters of the network in Fig. 68 have the following values:  $E = 100$  volts,  $T = 200$  microseconds,  $r = 5,000$  ohms,  $R = 15,000$  ohms,  $L = 0.1$  henry, and  $C = 0.002 \mu\text{f}$ .

- What is the maximum output voltage?
- At what instant during the generator pulse is the instantaneous output voltage equal to one-half the maximum output voltage? (This is the time delay of the network.)
- At what instant after the generator pulse disappears is the output voltage equal to one-half the maximum value?

**Prob. 2.** The network in Fig. 68 has the same parameters as in Prob. 1 with the exception of  $C$ , which is  $200 \mu\text{f}$ .

- What is the period of oscillation?
- What is the first instant that the output pulse reaches a value of 100 volts?
- After what time during the generator pulse is the amplitude of the transient term always less than 5 per cent of the steady-state term?

**Prob. 3.** The parameters of the network in Fig. 82 have the following values:  $E = 50$  volts,  $T = 30$  microseconds,  $r = 1,000$  ohms,  $R = 9,000$  ohms,  $L = 0.005$  henry, and  $C = 400 \mu\text{f}$ .

- What is the maximum positive output voltage?
- At what two instants is the network current a maximum? What is the difference of these two times?
- If the generator-pulse width is doubled, what is the maximum positive output voltage?

**Prob. 4.** In the network in Fig. 82,  $T > 1/M$ . Show for Cases 1, 2, and 3 that  $e_E = -e_0$  if the variable  $(t - T)$  in the equations for  $e_0$  is replaced by  $t$ .

**Prob. 5.** The parameters of the network in Fig. 94 have the following values:  $E = 40$  volts,  $T = 6$  microseconds,  $r = 500$  ohms,  $R = 3,500$  ohms,  $L = 0.001$  henry, and  $C = 390 \mu\text{f}$ .

- What is the maximum positive output voltage?
- What is the first time after  $t = 0$  that the output voltage is zero?
- At what instant during the generator pulse is the output voltage a maximum negative value, and what is this value?

**Prob. 6.** In the network in Fig. 82,  $C^*$  is replaced by a short circuit. Show that the output-voltage equations for Case 1 reduce to those of a simple  $RL$  network.

**Prob. 7.** The network in Fig. 94 is critically damped and the maximum negative output voltage during the 60-microsecond generator pulse is  $-8.1$  volts and occurs at  $t = 40$  microseconds. The inductance has a value of  $0.02$  henry. What are the values of  $(R + r)$ ,  $C$ , and  $E$ ?

**Prob. 8.** A critically damped series  $RLC$  network that is subjected to a 300-microsecond rectangular generator pulse has a value of  $M = 7,500 \text{ sec.}^{-1}$ , and  $L/R = 400$  microseconds.

- a. At what instant during the generator pulse is the voltage across  $L$  exactly equal to the voltage across  $R$ ?
- b. At what instant during the generator pulse is the voltage across  $L$  exactly equal to the negative of the voltage across  $R$ ?

## CHAPTER VI

### SERIES-PARALLEL NETWORKS CONTAINING RESISTANCE AND CAPACITANCE

Series networks have been analyzed in considerable detail in Chaps. III, IV, and V. In this chapter series-parallel networks containing resistance and capacitance only will be treated. Application of the very same principles used in series networks yields the pulse-response characteristics for series-parallel networks. Moreover, it will be demonstrated that many series-parallel networks can be regarded as series networks insofar as their pulse-response characteristics are concerned.

#### RC NETWORK WITH PARALLEL RC ACROSS OUTPUT

The first network to be considered is that in Fig. 105. The two-state analysis requires instantaneous-voltage equations for times during and after the pulse. Before proceeding, assume two branch currents  $i_R$  and  $i_C$  which equal  $dq_R/dt$  and  $dq_C/dt$ , respectively, and each of which flows through  $r$ .

**1. Equation for Output Pulse;**  
 $t = 0$  to  $t = T$ .—By utilizing Kirchhoff's law, three voltage equations can be written that apply during the time that the generator pulse exists. The equations are

$$E = r(i_{R_s} + i_{C_s}) + Ri_{R_s} \quad (110)$$

$$E = r(i_{R_s} + i_{C_s}) + \frac{q_{C_s}}{C} \quad (111)$$

$$Ri_{R_s} = \frac{q_{C_s}}{C} \quad (112)$$

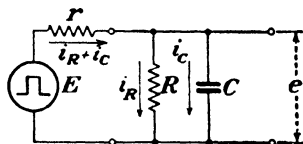


FIG. 105.—Series-parallel RC network with a rectangular-pulse generator.

Any two of these equations are independent, the third being dependent and derivable from the other two; hence, any two equations contain all the information.

There are two approaches toward determining the output voltage: (1) Find  $q_{C_E}$  as a function of time and divide by  $C$ , and (2) find  $i_{R_E}$  as a function of time and multiply by  $R$ . Both approaches lead to an identical result.

Suppose  $q_{C_E}$  as a function of time is sought first. Solve Eq. (112) for  $q_{C_E}$ .

$$q_{C_E} = RC i_{R_E}$$

Solve Eq. (110) for  $i_{R_E}$ .

$$i_{R_E} = \frac{E - r i_{C_E}}{R + r} \quad (113)$$

Substitute this value of  $i_{R_E}$  into the above equation for  $q_{C_E}$ .

$$q_{C_E} = \frac{RC}{R + r} (E - r i_{C_E}) \quad (114)$$

Since  $i_{C_E} = \frac{dq_{C_E}}{dt}$ , Eq. (114) becomes

$$q_{C_E} = \frac{RC}{R + r} \left( E - r \frac{dq_{C_E}}{dt} \right) \quad (114a)$$

This is a first-order differential equation soluble by separation of the variables  $q_{C_E}$  and  $t$ . Separate the variables.

$$\frac{dq_{C_E}}{E - \frac{RC}{R + r} q_{C_E}} = \frac{dt}{r}$$

Integrate to obtain the solution.

$$- \frac{RC}{R + r} \ln \left( E - \frac{R + r}{RC} q_{C_E} \right) = \frac{t}{r} + K_1 \quad (115)$$

To evaluate  $K_1$  consider that  $q_{C_E} = 0$  at  $t = 0$ .

$$K_1 = - \frac{RC}{R + r} \ln E$$

Insert this value of  $K_1$  into Eq. (115).

$$- \frac{RC}{R + r} \ln \left( \frac{E - \frac{R + r}{RC} q_{C_E}}{E} \right) = \frac{t}{r}$$

Convert to the exponential form.

$$E\epsilon^{-\frac{(R+r)t}{RC}} = E - \frac{R+r}{RC} q_{C_R}$$

Solve for  $q_{C_R}$ .

$$q_{C_R} = \frac{ERC}{R+r} [1 - \epsilon^{-\frac{(R+r)t}{RC}}]$$

For simplicity, define  $R_p \equiv Rr/(R+r)$  i.e., the equivalent resistance of  $R$  and  $r$  in parallel. This is not to be confused with physically placing  $R$  and  $r$  in parallel and is only a simplifying definition. The equation for  $q_{C_R}$  then becomes

$$q_{C_R} = \frac{ERC}{R+r} (1 - \epsilon^{-\frac{t}{R_p C}}) \quad (116)$$

The output voltage during the pulse interval is  $q_{C_R}/C$ .

$$e_E = \frac{ER}{R+r} (1 - \epsilon^{-\frac{t}{R_p C}}) \quad (117)$$

To check this result,  $i_{R_R}$  as a function of time will be found. A quick way of doing this is to find  $i_{C_R}$  from Eq. (116) and then to substitute into Eq. (113). Differentiate Eq. (116) to obtain  $i_{C_R}$ .

$$i_{C_R} = \frac{dq_{C_R}}{dt} = \frac{ERC\epsilon^{-\frac{t}{R_p C}}}{(R+r)R_p C} = \frac{E}{r}\epsilon^{-\frac{t}{R_p C}}$$

Substitute this value of  $i_{C_R}$  into Eq. (113).

$$i_{R_R} = \frac{E - r(E/r)\epsilon^{-\frac{t}{R_p C}}}{R+r} = \frac{E}{R+r} (1 - \epsilon^{-\frac{t}{R_p C}})$$

The output voltage is  $Ri_{R_R}$ , which is exactly the same as Eq. (117).

$$e_E = Ri_{R_R} = \frac{ER}{R+r} (1 - \epsilon^{-\frac{t}{R_p C}})$$

The steady-state term is  $ER/(R+r)$ , and the transient term is

$$- \frac{ER}{R+r} \epsilon^{-\frac{t}{R_p C}}.$$

**2. Equation for Output Pulse ;  $t = T$  to  $t = \infty$ .**—The output voltage that exists after the generator pulse has disappeared can be obtained after setting  $E$  equal to zero in the original differential equations.

$$0 = r(i_{R_0} + i_{C_0}) + Ri_{R_0} \quad (110a)$$

$$Ri_{R_0} = \frac{qc_0}{C} \quad (112a)$$

A solution for either  $qc_0$  or  $i_{R_0}$  will lead directly to the equation for the output voltage.

To solve for  $qc_0$ , first solve Eq. (112a) for  $i_{R_0}$ .

$$i_{R_0} = \frac{qc_0}{RC} \quad (118)$$

Substitute this value of  $i_{R_0}$  into Eq. (110a), replace  $i_{C_0}$  by its equal  $dqc_0/dt$ , and simplify the result by utilizing the definition  $R_P \equiv Rr/(R + r)$ .

$$qc_0 = -R_P C \frac{dq_{C_0}}{dt}$$

Separate variables and integrate to solve for  $q_{C_0}$  as a function of time.

$$\ln q_{C_0} = -\frac{t}{R_P C} + K_2 \quad (119)$$

The value of charge at  $t = T$  can be found from Eq. (116) and leads to an evaluation of  $K_2$ .

$$(q_{C_0})_{t=T} = \frac{ERC}{R + r} (1 - e^{-\frac{T}{R_P C}}) \quad (116a)$$

$$K_2 = \ln \left[ \frac{ERC}{R + r} (1 - e^{-\frac{T}{R_P C}}) \right] + \frac{T}{R_P C}$$

Substitute  $K_2$  into Eq. (119) and convert to the exponential form.

$$q_{C_0} = \frac{ERC}{R + r} (e^{\frac{T}{R_P C}} - 1) e^{-\frac{t}{R_P C}} \quad (120)$$

The output voltage for the time after the generator pulse becomes zero is  $q_{C_0}/C$ .

$$e_0 = \frac{ER}{R + r} (e^{\frac{T}{R_P C}} - 1) e^{-\frac{t}{R_P C}} \quad (121)$$

It is apparent from this equation that the steady-state term is zero.

For completeness, the expression for  $i_{R_0}$  can be found, and Eq. (121) can be checked. Eliminate  $q_c$  from Eqs. (118) and (120).

$$i_{R_0} = \frac{E}{R + r} (\epsilon^{\frac{T}{RrC}} - 1) \epsilon^{-\frac{t}{RrC}}$$

The output voltage is  $Ri_{R_0}$  and agrees identically with Eq. (121).

**3. Familiarization with Output-pulse Equations.**—To help grasp the significance of Eqs. (117) and (121) it is beneficial to become familiar with them by means of some examples.

*Example 1.*  $R = \infty$ .—If  $R$  is removed from the network ( $R = \infty$ ), the equations for  $e$  should be<sup>1</sup>

$$e_E = E(1 - \epsilon^{-\frac{t}{rC}}) \quad (33')$$

$$e_0 = E(\epsilon^{\frac{T}{rC}} - 1) \epsilon^{-\frac{t}{rC}} \quad (36')$$

To find out if Eqs. (117) and (121) contain this solution, first recognize that  $R_P = \frac{Rr}{R + r} = \frac{r}{1 + (r/R)}$  is equal to  $r$ , if  $R$  is infinite. Upon substituting  $R_P = r$ , Eq. (117) immediately becomes

$$e_E = E(1 - \epsilon^{-\frac{t}{rC}})$$

and Eq. (121) becomes

$$e_0 = E(\epsilon^{\frac{T}{rC}} - 1) \epsilon^{-\frac{t}{rC}}$$

Therefore, Eqs. (117) and (121) contain the special solution for the case where  $R$  is infinite.

*Example 2.*  $C = 0$ .—If  $C$  is removed from the network ( $C = 0$ ), the equations for the output voltage must be

$$e_E = \frac{Rr}{R + r}$$

$$e_0 = 0$$

<sup>1</sup> See Chap. III, Eqs. (33) and (36). If  $(R + r)$  is set equal to  $r$ , the above equations result.

since the network now consists of the voltage divider  $r$  and  $R$ . Set  $C$  equal to zero in Eqs. (117) and (121).

$$e_E = \frac{ER}{R+r} (1 - e^{-\infty}) = \frac{ER}{R+r} (1 - 0) = \frac{ER}{R+r}$$

$$e_0 = \frac{ER}{R+r} (\epsilon^{R_p C} - 1) \epsilon^{-R_p C} = \frac{ER}{R+r} (1 - \epsilon^{-\frac{T}{R_p C}}) \epsilon^{-\frac{(t-T)}{R_p C}}$$

$$= \frac{ER}{R+r} (1 - e^{-\infty}) e^{-\infty} = \frac{ER}{R+r} (1 - 0) 0 = 0$$

Again Eqs. (117) and (121) contain the special solution.

**Example 3.**— $r = 0$ . If the internal resistance of the generator is zero, then  $e_E = E$  and  $e_0 = 0$  since the output voltage is directly across the generator in this instance. If  $r$  is set equal to zero, then  $R_p = Rr/(R+r) = 0$ . Equation (117) becomes

$$e_E = \frac{ER}{R+0} (1 - e^{-\infty}) = \frac{ER}{R} (1 - 0) = E$$

and Eq. (121) becomes

$$e_0 = \frac{ER}{R+0} (1 - e^{-\infty}) e^{-\infty} = E(1 - 0) 0 = 0$$

These special examples are indicative of the fact that Eqs. (117) and (121) are general solutions for the instantaneous output voltage.

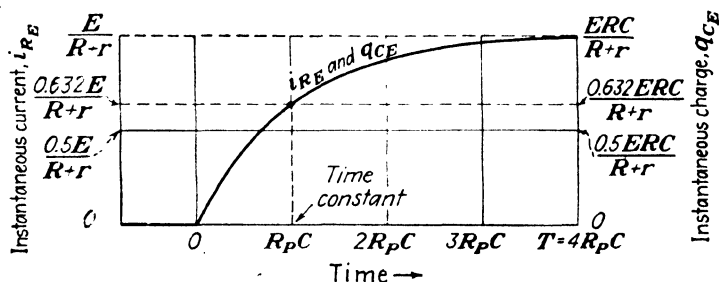
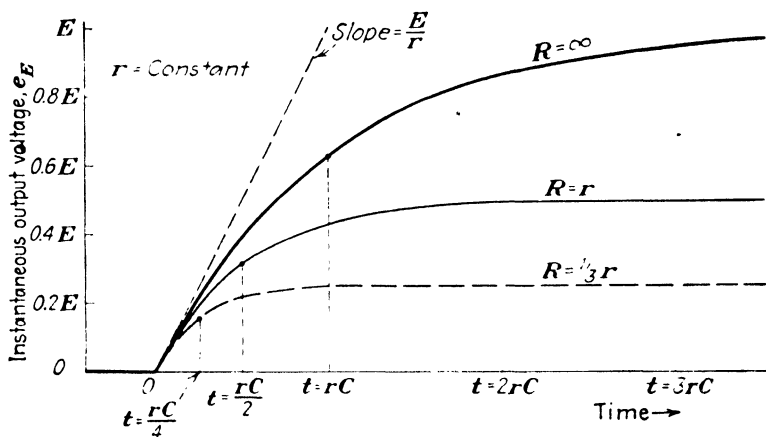


FIG. 106.—Exponential growth of current through  $R$  and charge on  $C$  for the network in Fig. 105 during the generator pulse.

**4. Network Behavior;  $t = 0$  to  $t = T$ .**—While it is important to have the mathematical solution for the network, a graphical picture is also very helpful. The instantaneous current flowing through  $R$  and the instantaneous charge on  $C$  determine the output-voltage behavior. In Fig. 106 the growth of current through

$R$  and growth of charge on  $C$  during the pulse interval are shown for a generator-pulse width that is greater than the time constant  $R_pC$ . There are several properties of this curve that help to understand what is actually occurring in the network during the generator pulse. First,  $q_{c_R}$  and  $i_{R_R}$  are identical in shape. This can be explained by realizing three things: (1) The voltage across  $C$  is directly proportional to  $q_{c_R}$ , (2) the voltage across  $R$  is directly proportional to  $i_{R_R}$ , and (3) the voltage across  $C$  and across  $R$  is always identical. Hence, the identity in shape of  $i_{R_R}$  and  $q_{c_R}$  can be formulated by the simple mathematical relations

- (1)  $e_{c_R} = \frac{q_{c_R}}{C}$
- (2)  $e_{R_R} = Ri_{R_R}$
- (3)  $e_C = e_E$



\* FIG. 107. — Output voltage during the generator pulse of the network in Fig. 105 for three values of  $R$ . The time constant is reduced when  $R$  is made smaller, but the output voltage is also decreased.

It can be deduced that  $Ri_{R_R} = q_{c_R}/C$ , which is the same as Eq. (112). Therefore,  $i_{R_R}$  and  $q_{c_R}$  vary in exactly the same manner with time because  $R$  and  $C$  are constant.

The second pertinent point about Fig. 106 is that the steady-state value of  $i_{R_R}$  is  $E/(R + r)$ . Referring to the network, this means that the flow of charge on  $C$  must be zero and the voltage across  $C$  must be at its maximum possible value when  $i_{R_R}$  equals its steady-state value. This maximum value, which must equal the voltage across  $R$ ,  $ER/(R + r)$ , is always less than the

generator-pulse height  $E$ . From the standpoint of maximum value of  $q_{C\max}$ , the same conclusion is reached; namely,

$$e_{E\max} = \frac{q_{C\max}}{C} = \frac{ER}{R+r}$$

The third point is that the build-up of charge and of current is in accordance with a time constant equal to  $R_P C$ .  $R_P$  is of course smaller than either  $r$  or  $R$ , so the time constant is smaller than  $rC$  or  $RC$ . If  $R$  were removed from the network,  $C$  would attain a greater charge, the output voltage would have a steady-state value equal to  $E$ , and the time constant would be  $rC$ . Thus it is seen that the addition of a resistance  $R$  across  $C$  reduces the time constant at the sacrifice of output voltage. Refer to Fig. 107.

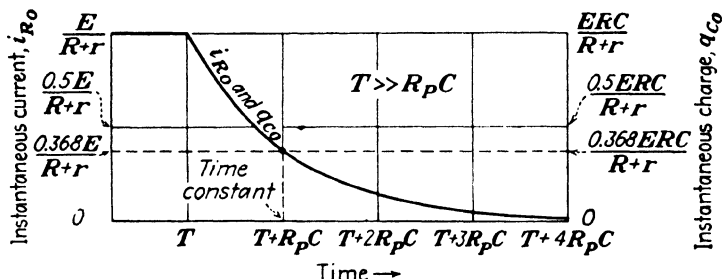


FIG. 108.—Exponential decay of current through  $R$  and charge on  $C$  for the network in Fig. 105 after a generator pulse that is long compared with the time constant has disappeared.

**5. Network Behavior;  $t = T$  to  $t = \infty$ .**—Suppose that a generator pulse of very long duration compared with  $R_P C$  has been applied to the network. At the instant the pulse disappears, the output voltage will be

$$e_{Er} = \frac{ER}{R+r} = e_{0r}$$

from Eq. (117) or (121) where it is assumed  $t = T$  and  $T \gg R_P C$ . The instantaneous charge on  $C$  will be

$$q_{Cr} = e_{0r}C = \frac{ERC}{R+r}$$

and the instantaneous current through  $R$  will be

$$i_{Rr} = \frac{e_{0r}}{R} = \frac{E}{R+r}$$

Figure 108 shows the behavior of  $q_c$ , and  $i_R$ , from the time  $T$  on. The decay is in accordance with a time constant that again equals  $R_P C$ .

It is interesting to note that  $e$  remains positive at all times. This can be explained in two ways. The most direct explanation is that  $C$ , which has been charged to a certain positive voltage by the pulse, simply discharges through  $R$  and  $r$  in parallel and suffers no reversal of polarity. Another explanation on the basis of current flow through  $R$  uncovers an interesting point; namely, current through  $R$  is always unidirectional. The generator pulse causes current to flow through  $R$  in the same direction as current flow caused by the discharge of  $C$  through  $R$ .

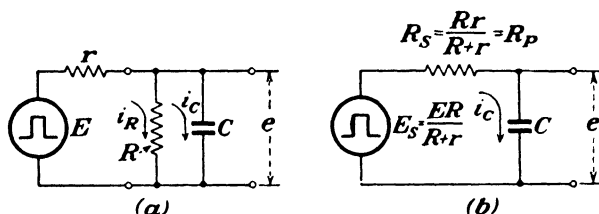


FIG. 109.—A series-parallel network and a series network that produce exactly the same output voltage.

**6. Equivalent Series Network.**—Frequently, it is useful to correlate new networks with familiar ones by transforming the new network into an equivalent network that resembles the familiar one. Such a correlation can be made in this case. On reexamination of the network behavior, the following properties are evident:

1. The charging time constant is  $R_P C$ .
2. The maximum output voltage is  $ER/(R + r)$  if the pulse width is large compared with  $R_P C$ .
3. The output pulse is always positive and rises and falls exponentially.
4. The discharging time constant is  $R_P C$ .

This output-voltage behavior is very similar to that of a simple series  $RC$  network, or that of a series  $RL$  network, and suggests that there may be a series  $RC$  or series  $RL$  equivalent network for this series-parallel network. The equivalent series  $RC$  network is shown in Fig. 109b. Insofar as the output voltage is concerned, the networks in Fig. 109 are identical. Each of

the four properties listed applies equally well to both networks. Thus the behavior of output voltage in the series-parallel network is exactly the same as the output-voltage behavior in the equivalent series network.

Application of Thévenin's theorem, which was introduced in Chap. II, also leads to this same equivalent network. To show how the theorem can be applied, refer to Fig. 110. In this figure, the network of Fig. 109a has been redrawn but is essentially unaltered. The equivalent series network can be found by Thévenin's theorem when  $C$  is removed. The voltage

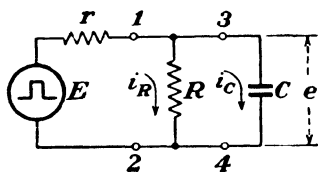


FIG. 110.—Series-parallel  $RC$  network which is the same as that in Fig. 105.

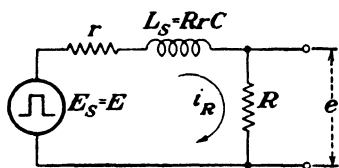


FIG. 111.—A series network that produces exactly the same output voltage as the series-parallel network in Fig. 109a.

that appears across terminals 3-4 with  $C$  removed is the equivalent generator voltage.

$$E_s = \frac{ER}{R + r}$$

The resistance across terminals 3-4, with  $C$  removed and the generator short-circuited, is  $R$  in parallel with  $r$  and is the equivalent series resistance.

$$R_s = \frac{Rr}{R + r} \equiv R_p$$

Thus the network in Fig. 109b results from Thévenin's theorem as well as from an analogy based upon a complete solution for the output voltage. Recall that the network in Fig. 109b is equivalent only as far as the output terminals are concerned, and that it is not equivalent as far as the generator is concerned.

Instead of having the capacitance  $C$  across the output of the series equivalent network, suppose the resistance  $R$  is desired across the output. The equivalent network will then be the series  $RL$  network in Fig. 111. Inspection of this equivalent

network reveals that the same four properties listed for the series-parallel network are equally applicable to this network. The time constant is  $L_s/(R + r) = RrC/(R + r) = R_pC$ , which is the same as the time constant in the network shown in Fig. 109. [The dimension of (resistance)<sup>2</sup>  $\times$  capacitance is inductance, as was demonstrated in Chap. I.] The output voltage rises exponentially during the generator pulse and has a steady-state value equal to  $ER/(R + r)$ . The output voltage is always positive and decreases exponentially after the generator pulse has disappeared.

**7. Equivalent-series-network Method.**—This equivalent-network viewpoint might well be examined more closely. If other series-parallel networks have equivalent series networks,

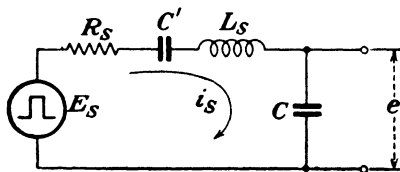


FIG. 112.—General series  $RLC'$  network with capacitance output.

then the pulse-response characteristics can be quickly predicted from the work of the preceding chapters. The two equivalent networks in Figs. 109*b* and 111 have been derived more or less intuitively on the basis of an analogy between the complete analysis of the series-parallel network and the complete analysis of series networks. A search for a precise method by which the equivalent series network can be found shows that one need proceed no further than the differential equations for the series-parallel network. In other words, a solution of the differential equations is not necessary to find the equivalent series network.

Suppose an equivalent series network in which the output voltage appears across  $C$  only is desired for the network in Fig. 105. Such a network is shown in general in Fig. 112. The network in Fig. 112 will be equivalent only if the behavior of charge on  $C$  in the equivalent network is exactly the same as the behavior of charge on  $C$  in the actual network. The differential equation for charge on  $C$  in the actual network and during the generator pulse is given by Eq. (114*a*). The problem thus resolves itself into finding the values of  $E_s$ ,  $L_s$ ,  $R_s$ , and  $C'$  that give

the behavior of charge on  $C$  required by Eq. (114a). Rewrite Eq. (114a).

$$\frac{ER}{R+r} = \frac{Rr}{R+r} \frac{dq_{c_s}}{dt} + \frac{q_{c_s}}{C} \quad (114b)$$

The differential equation for the equivalent series network in Fig. 112 is

$$E_s = L_s \frac{d^2 q_s}{dt^2} + R_s \frac{dq_s}{dt} + \left( \frac{1}{C} + \frac{1}{C'} \right) q_s \quad (122)$$

It is only necessary to equate the coefficients of the terms in Eq. (114b) to the same terms in Eq. (122) to determine the values of  $E_s$ ,  $L_s$ ,  $R_s$ , and  $C'$  that will make  $q_s$  in Eq. (122) the same as  $q_{c_s}$  in Eq. (114b). Equate the coefficients of like terms.

$$\begin{aligned} E_s &= \frac{ER}{R+r} \\ L_s &= 0 \\ R_s &= \frac{Rr}{R+r} \equiv R_p \\ \frac{1}{C} + \frac{1}{C'} &= \frac{1}{C} \quad \text{or} \quad C' = \infty \end{aligned}$$

With these values, Eq. (122) becomes identical with Eq. (114b); *i.e.*,  $q_s \equiv q_{c_s}$ , and the required equivalent series network is that shown in Fig. 109b.

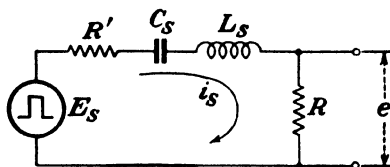


FIG. 113.—General series  $RLC$  network with resistance output.

By this method of equating coefficients of like terms, it is informative to verify the equivalent series network for the case where  $R$  only appears across the output. A general series network with resistance output is shown in Fig. 113, and the differential equation that applies during the generator pulse is

$$E_s = L_s \frac{di_s}{dt} + (R' + R)i_s + \frac{q_s}{C_s} \quad (123)$$

The differential equation for current through  $R$  in the actual network can be obtained by differentiating Eq. (112) and inserting the value of  $dq_{C_s}/dt = i_{C_s}$  into Eq. (110) with the following result:

$$E = r \left( i_{R_s} + RC \frac{di_{R_s}}{dt} \right) + Ri_{R_s} = RrC \frac{di_{R_s}}{dt} + (R + r)i_{R_s}$$

Equate the coefficients of this equation to the coefficients of like terms in Eq. (123).

$$\begin{aligned} E_s &= E \\ L_s &= RrC \\ R + r &= R' + R \quad \text{or} \quad R' = r \\ 0 &= \frac{1}{C_s} \quad \text{or} \quad C_s = \infty \end{aligned}$$

Substitution of these values into Eq. (123) requires the current  $i_s$  in the equivalent network to be the same as  $i_{R_s}$ . These equivalent-network values give the same network that was obtained by analogy in Fig. 111.

Strictly speaking, it has been shown so far that the series network is equivalent only during the existence of the generator pulse. To prove that the same equivalent series network applies after the generator pulse disappears is quite simple. The differential equation for the series-parallel network after the generator pulse disappears can be written by setting  $E = 0$ . When this is done, equating coefficients indicates that the equivalent generator voltage likewise becomes zero. This involves no change in the other coefficients of the differential equation, and therefore the resistance, inductance, and capacitance of the equivalent series network are unaffected. In other words, the same equivalent network applies both during and after the generator pulse, and the equivalent generator voltage is a rectangular pulse.

The two foregoing equivalent series networks show that a given series-parallel network does not have a unique equivalent series network. This is often true. In fact, it is possible to have more than one equivalent series network even in the case where the element across which the output voltage appears is the same. An example of this is given in Fig. 119a.

Since this method enables the pulse-response characteristic

to be determined without solving the differential equation, it is expedient to use it wherever possible. It is now becoming evident that the analysis of series networks is a very valuable tool that can be used to analyze series-parallel networks. Although the equivalent series network is very useful, the action within the physical network that it replaces is obscured. Usually the only part of the actual network that can be preserved is the element across which the output pulse appears. This is no great loss, however, because the pulse-response characteristic is of chief interest.

### RC NETWORKS WITH CAPACITANCE ACROSS OUTPUT

There is an unlimited variety of series-parallel networks containing resistance and capacitance that have a capacitance

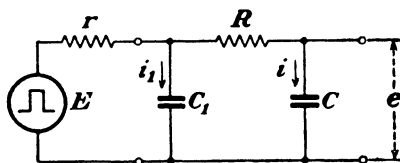


FIG. 114.—Series-parallel  $RC$  network with rectangular-pulse generator.

across the output. In this section, two specific networks will be analyzed for their pulse-response characteristics by the equivalent-series-network method. Then the limitations of the method will be pointed out. When it is specific that a capacitance is connected across the output, this does not preclude the possibility of a shunt resistance across the capacitance, even though the two examples to be worked out have capacitance only across the output of the series-parallel network. The network of Fig. 105, for instance, can be considered to be in this class. However, the equivalent series network will have a capacitance *only* across the output.

**8. Pulse-response Characteristic. Example 1.**—The first network to be analyzed is shown in Fig. 114, where the two branch currents are assumed to be  $i_1$  and  $i$  corresponding to the rate of change of branch charges,  $q_1$  and  $q$ . The differential equation for  $q$  in terms of  $t$  only will enable the parameters of an equivalent series network to be determined and therefore will lead to the pulse-response characteristic in terms of a

known series network Two independent equations for instantaneous voltage during the pulse for the network in Fig. 114 are

$$E = r(i_1 + i) + Ri + \frac{q}{C} \quad (124)$$

$$\frac{q_1}{C_1} = Ri + \frac{q}{C} \quad (125)$$

Differentiation of Eq. (125) yields the value of  $i_1$ .

$$i_1 = \frac{dq_1}{dt} = C_1 R \frac{di}{dt} + \frac{C_1}{C} \frac{dq}{dt}$$

Substitution of this value of  $i_1$  into Eq. (124) results in the differential equation for  $q$ .

$$E = r \left( C_1 R \frac{di}{dt} + \frac{C_1}{C} \frac{dq}{dt} + i \right) + Ri + \frac{q}{C}$$

Expand and collect terms.

$$E = RrC_1 \frac{d^2q}{dt^2} + \left( R + r + \frac{C_1}{C} r \right) \frac{dq}{dt} + \frac{q}{C}$$

The equivalent series network consequently contains the parameters

$$E_s = E$$

$$L_s = RrC_1$$

$$R_s = R + r + \frac{C_1}{C} r$$

$$C_s = C$$

The equivalent network is given in Fig. 115. A similar network has been studied in detail in Chap. V. To determine

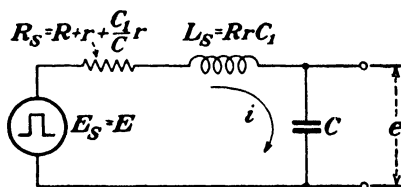


FIG. 115.—Equivalent series network for that in Fig. 114.

completely the pulse-response characteristics, it is necessary to find out whether  $R_s$  is greater than, equal to, or less than  $2\sqrt{L_s/C}$ . Before showing that  $R_s$  is always greater than  $2\sqrt{L_s/C}$ , which is the overdamped or nonoscillatory case, it is

helpful to examine the original network from an energy viewpoint. In Fig. 114 it is seen that only electrostatic energy can be stored since there is no inductance in the network. Therefore, it is impossible for oscillation to take place inasmuch as an exchange of energy is required for oscillation. One might deduce from this consideration that the network is nonoscillatory. To show this rigorously, it is necessary to prove that  $R_s > 2\sqrt{L_s/C}$ , or  $R_s^2 > 4L_s/C$ , for all values of  $R$ ,  $r$ ,  $C$ , and  $C_1$  that are greater than zero. A proof of this follows:

$$R_s^2 = \left(R + r + \frac{C_1}{C}r\right)^2 = (R + r)^2 + 2\frac{C_1}{C}r(R + r) + r^2\left(\frac{C_1}{C}\right)^2$$

$$\frac{4L_s}{C} = \frac{4RrC_1}{C}$$

By inspection it is necessary to prove that

$$r^2\left(\frac{C_1}{C}\right)^2 + (2r^2 - 2Rr)\frac{C_1}{C} + (R + r)^2 > 0$$

The term on the left can be considered to be a function of  $C_1/C$  that must be shown to be positive (greater than zero) for all positive values of  $R$ ,  $r$ , and  $C_1/C$ .

$$f\left(\frac{C_1}{C}\right) = r^2\left(\frac{C_1}{C}\right)^2 + (2r^2 - 2Rr)\frac{C_1}{C} + (R + r)^2 > 0$$

$f(C_1/C)$  is positive for  $C_1/C = 0$ , and if this function has no positive real roots it will be positive for all values of  $C_1/C$ . In other words, if  $f(C_1/C)$  were graphed against  $C_1/C$ , it would have a positive value at  $C_1/C = 0$ . The question is: Does the graph of  $f(C_1/C)$  cross zero at some positive value of  $C_1/C$ ; i.e., does it have any positive real roots? If it does, this would signify that  $f(C_1/C)$  has negative values and hence could be less than zero, which is contrary to the above inequality. Thus the proof resolves itself into demonstrating that  $f(C_1/C)$  can have no positive real roots for all positive values of  $R$ ,  $r$ , and  $C_1/C$ .

From the quadratic formula,  $f(C_1/C)$  will have no positive real roots if  $\sqrt{(2r^2 - 2Rr)^2 - 4r^2(R + r)^2}$  is imaginary. Expand and cancel terms.

$$\sqrt{4r^2 - 8Rr^3 + 4R^2r^2 - 4R^2r^2 - 8Rr^3 - 4r^4} = 4r\sqrt{-Rr}$$

This is imaginary for all positive values of  $R$  and  $r$ . Therefore,  $f(C_1/C)$  has no positive real roots when  $R$  and  $r$  are positive and

$$f\left(\frac{C_1}{C}\right) > 0$$

for all positive values of  $R$ ,  $r$ , and  $C_1/C$ . Thus the oscillatory case is impossible.

The critically damped case, as well as the oscillatory case, is also impossible in the network in Fig. 115 for all positive values of  $R$ ,  $r$ , and  $C_1/C$ . This can be demonstrated by setting  $R_s^2$  equal to  $4L_s/C$ . When this is done, the following equation results:

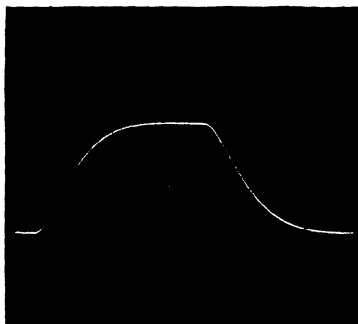
$$r^2 \left(\frac{C_1}{C}\right)^2 + (2r^2 - 2Rr) \frac{C_1}{C} + (R + r)^2 = 0 = f\left(\frac{C_1}{C}\right)$$

Solve for  $C_1/C$ .

$$\frac{C_1}{C} = \frac{2Rr - 2r^2 \pm 4r \sqrt{-Rr}}{2r^2} = \frac{R}{r} - 1 \pm \frac{2}{r} \sqrt{-Rr}$$

This shows that  $C_1/C$  is imaginary for all positive values of  $R$  and  $r$ . However,  $C_1/C$  must be real and therefore the critically damped case is impossible.

The oscillogram in Fig. 116 is an illustration of the output voltage obtained for the network in Fig. 114. The output voltage is overdamped. The same voltage could be obtained across the capacitance in an equivalent overdamped series  $RLC$  network.



### 9. Pulse-response Characteristic.

**Example 2.**—The second specific network to be analyzed is that in Fig. 117. Again it is possible to find the differential equation for  $q$  as a function of time only and to obtain an equivalent series network. Two

FIG. 116.—Output pulse of the network in Fig. 114 when  $T \gg 1/M_s$ . Note the resemblance between this pulse and that in Fig. 77.

independent equations for instantaneous voltage during the pulse are

$$E = r(i_1 + i) + \frac{q}{C_2} + Ri + \frac{q}{C} \quad (126)$$

$$R_1 i_1 + \frac{q_1}{C_1} = \frac{q}{C_2} + Ri + \frac{q}{C} \quad (127)$$

In this instance, obtaining  $q$  as a function of time is slightly more cumbersome than before. First, Eq. (127) can be solved for  $q_1$ .

$$q_1 = \frac{C_1}{C_2} q + C_1 Ri + \frac{C_1}{C} q - C_1 R_1 i_1$$

Then Eq. (126) is solved for  $i_1$ .

$$i_1 = \frac{E - (R + r)i - (q/C_2) - (q/C)}{r}$$

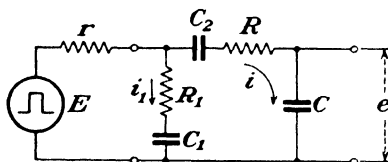


FIG. 117. Series-parallel network with rectangular-pulse generator.

Now if  $q_1$  is differentiated,

$$\frac{dq_1}{dt} = i_1 = \frac{C_1}{C_2} \frac{dq}{dt} + C_1 R \frac{di}{dt} + \frac{C_1}{C} \frac{dq}{dt} - C_1 R_1 \frac{di_1}{dt}$$

and the value of  $i_1$  is substituted, an equation involving only  $q$  and  $i$  results.

$$\frac{E - (R + r)i - \frac{q}{C_2} - \frac{q}{C}}{r} = \frac{C_1}{C_2} \frac{dq}{dt} + C_1 R \frac{di}{dt} + \frac{C_1}{C} \frac{dq}{dt} - C_1 R_1 \left[ \frac{-(R + r) \frac{di}{dt} - \frac{1}{C_2} \frac{dq}{dt} - \frac{1}{C} \frac{dq}{dt}}{r} \right]$$

Rewrite this equation in terms of  $q$  only.

$$E = [RrC_1 + R_1C_1(R + r)] \frac{d^2q}{dt^2} + \left[ R + r + (R_1C_1 + rC_1) \left( \frac{1}{C_2} + \frac{1}{C} \right) \right] \frac{dq}{dt} + \left( \frac{1}{C_2} + \frac{1}{C} \right) q$$

The equivalent-series-network parameters are determined from the coefficients of this equation. The equivalent network is shown in Fig. 118. This is also an overdamped network, as are all equivalent networks for actual networks containing  $R$  and  $C$  only. Chapter V gives the complete pulse-response characteristics for this type of series network.

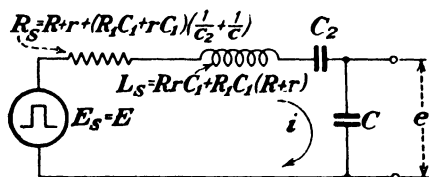


FIG. 118.—Equivalent series network for that in Fig. 117.

**10. Limitations on Equivalent-series-network Method.**—The parameters of the equivalent series network are obtained by equating coefficients of terms in the general series network differential equation to like terms in the series-parallel network differential equation. There is an obvious limitation on this method that can make it impossible to obtain an equivalent series network. It occurs when the series-parallel network differential equation is of higher order than the general series-network differential equation. For example, if a third-order differential equation for  $q$  arose from a series-parallel network, there would be no third-order term in the general series-network differential equation for equating coefficients. Another way of stating the limitation is this: No linear parameter exists that, when placed in a linear series network, gives rise to a differential equation for  $q$  that is of higher order than the second, or to a differential equation for  $i$  that is of higher order than the first. Since it is possible to obtain third and higher order differential equations from series-parallel networks, evidently series-parallel networks can be more complex than series networks. When an equivalent series network does not exist, it becomes necessary to solve completely the higher order differential equation to determine the pulse-response characteristics.

A particular limitation on equivalent series networks with capacitance output is that the steady-state term during the pulse must be different from zero. In some series-parallel

networks, the steady-state term during the pulse is zero; for instance, if a series capacitor is inserted between  $r$  and  $R$  in the network of Fig. 105. This limitation exists because the steady-state value of voltage across  $C$  during the generator pulse is always different from zero in *any* series network. To overcome such a situation it becomes necessary to use either a resistance or an inductance across the output of the equivalent series network. This will be demonstrated in the next section.

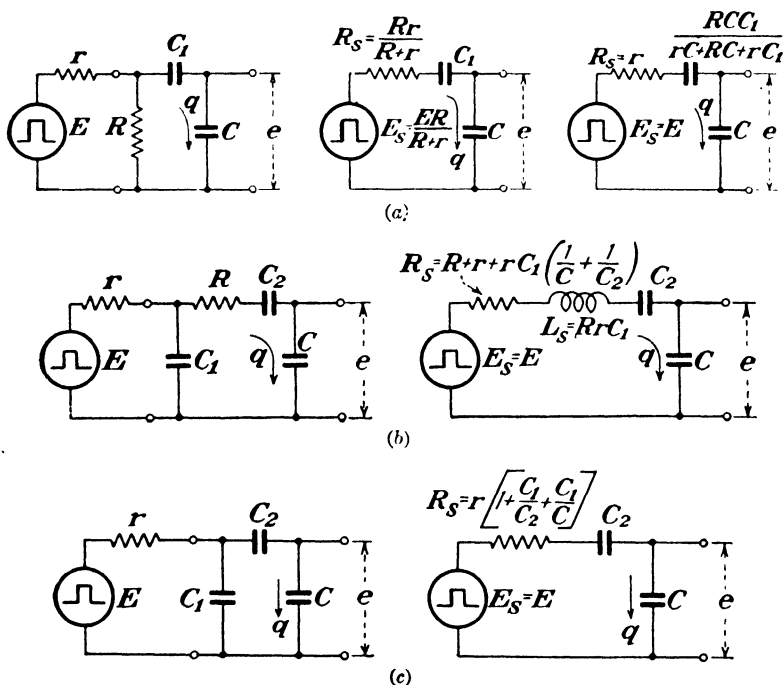


FIG. 119.—Examples of equivalent networks with capacitance output.

Figure 119 shows several series-parallel networks with capacitance across the output and their corresponding equivalent series networks.

#### RC NETWORKS WITH RESISTANCE ACROSS OUTPUT

In this section series-parallel networks that have a resistance across the output in both the actual and equivalent networks will be treated. One such network (Fig. 105) has already been

analyzed. Two additional examples will be given. The first example demonstrates the method for a more complicated network than that in Fig. 105. The second example shows how an equivalent series network with resistance across the output can be found in a case where it is impossible to find one with capacitance across the output.

**11. Pulse-response Characteristic. Example 1.**—Figure 120 shows the series-parallel network to be analyzed. The pulse-

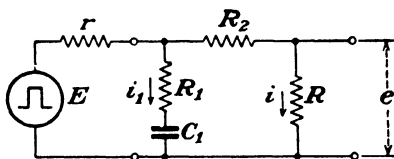


FIG. 120.—Series-parallel RC network with rectangular-pulse generator.

response characteristic can be found indirectly by finding the equivalent series network. To do this, the differential equation for  $i$ , the instantaneous current through  $R$ , in terms of  $t$  only is required. This can be determined from the equations

$$E = r(i_1 + i) + (R_2 + R)i \quad (128)$$

$$R_1 i_1 + \frac{q_1}{C_1} = (R_2 + R)i \quad (129)$$

The differential equation that results after  $q_1$  and  $i_1 = dq_1/dt$

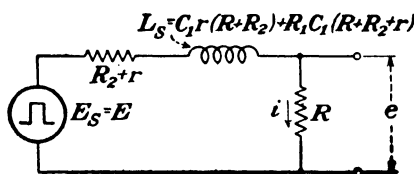


FIG. 121. Equivalent series network for that in Fig. 120.<sup>1</sup>

are eliminated from Eqs. (128) and (129) is the required equation for  $i$ .

$$E = [C_1 r (R + R_2) + R_1 C_1 (R + R_2 + r)] \frac{di}{dt} + (R + R_2 + r)i$$

The equivalent series network is therefore determined and is given in Fig. 121. Refer to Chap. IV for the pulse-response characteristic of this network.

**12. Pulse-response Characteristic. Example 2.**—The network in Fig. 122 provides an opportunity to study a case where an equivalent series network might be found with either a

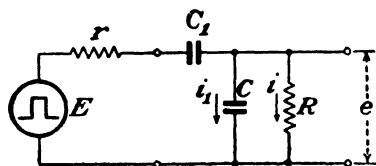


FIG. 122.—Series-parallel RC network with a rectangular-pulse generator.

resistance or capacitance across the output. A brief examination of the network reveals that the steady-state output voltage during the generator pulse is zero due to the presence of  $C_1$ . This means that an equivalent series network with  $C$  across the output is not possible as was explained on page 184. Nevertheless, if an attempt is made to find an equivalent network with  $C$  across the output, it will be possible to discover how the method fails. Two independent voltage equations are

$$E = r(i_1 + i) + \frac{q}{C_1} + \frac{q_1}{C_1} + Ri \quad (130)$$

$$\frac{q_1}{C} = Ri \quad (131)$$

The customary procedure is to eliminate  $q$  and  $i$  from these equations in order to obtain the differential equation for  $q_1$  in terms of  $t$  only. Solve Eq. (131) for  $i$ .

$$i = \frac{q_1}{RC} = \frac{dq}{dt}$$

Separate variables and integrate.

$$q = \frac{1}{RC} \int q_1 dt$$

Substitute these values of  $i$  and  $q$  into Eq. (130).

$$E = r \left( i_1 + \frac{q_1}{RC} \right) + \frac{(1/RC) \int q_1 dt}{C_1} + \frac{q_1}{C_1} + \frac{q_1}{C}$$

Collect like terms.

$$E = r \frac{dq_1}{dt} + \left( \frac{r}{RC} + \frac{1}{C_1} + \frac{1}{C} \right) q_1 + \frac{1}{RCC_1} \int q_1 dt$$

Now the last term in this differential equation prohibits an equivalent series network because no linear element placed in a

linear series network develops a voltage that is proportional to the time integral of the charge. This point will be discussed in more detail in Chap. VII.

However, an equivalent series network in which the output appears across  $R$  can be found, because in a series network the steady-state value of voltage across a resistance can be

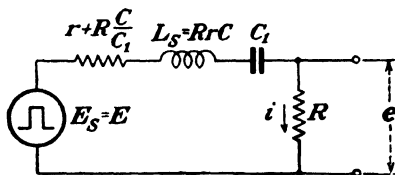


FIG. 123.—Equivalent series network for that in Fig. 122.

zero. To find the equivalent network it is necessary to obtain the differential equation for  $i$  in terms of  $t$  only. Solve Eq. (131) for  $q_1$  and differentiate to find  $i_1$ .

$$q_1 = RCi; \quad i_1 = \frac{dq_1}{dt} = RC \frac{di}{dt}$$

Substitute these values of  $q_1$  and  $i_1$  into Eq. (130).

$$E = r \left( RC \frac{di}{dt} + i \right) + \frac{q}{C_1} + \frac{RC}{C_1} i + Ri$$

Collect like terms.

$$E = RrC \frac{di}{dt} + \left( r + R \frac{C}{C_1} + R \right) i + \frac{q}{C_1}$$

The equivalent series network, Fig. 123, is determined from the coefficients of this equation. It is evident from the equivalent network that the steady-state value of voltage during the generator pulse is zero.

To show that the network is overdamped for all positive values of  $R$ ,  $r$ ,  $C$ , and  $C_1$ , it must be demonstrated that

$$\left( r + R \frac{C}{C_1} + R \right) > 2 \sqrt{\frac{RrC}{C_1}}$$

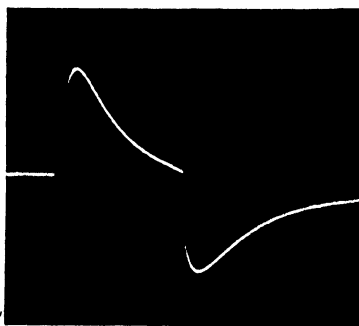


FIG. 124.—Output pulse of the network in Fig. 122 when  $T \gg 1/M_S$ . Note the resemblance between this pulse and that in Fig. 89.

This proof is very similar to the one given on page 180. The oscillogram in Fig. 124 is an illustration of the overdamped output voltage that is obtained for the network in Fig. 122. The same voltage is produced across  $R$  in an overdamped series  $RLC$  network.

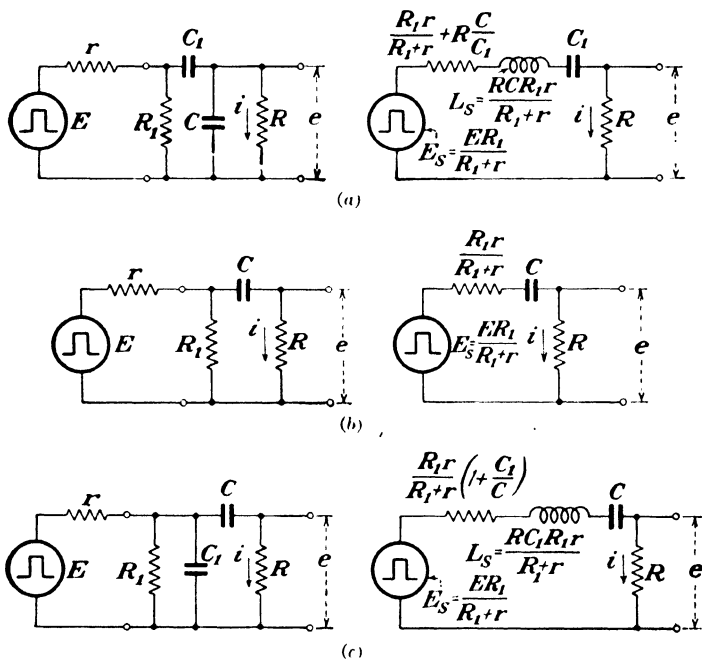


FIG. 125.—Examples of equivalent networks with resistance output.

Figure 125 shows several series-parallel  $RC$  networks accompanied by their equivalent series networks. It is suggested that these networks be verified to acquire experience in solving for the equivalent series network.

### CONCLUSION

There are many series-parallel  $RC$  networks that occur in practice for which equivalent series networks can be found. In such instances, the equivalent-network viewpoint can be very helpful in analyzing the transient response of the network. In other words, the use of the classical method is not restricted to series networks only. If no series equivalent network is

possible, which is the case when a third or higher order differential equation for  $q$  results, the classical method becomes quite involved. There is not only the matter of solving a higher order differential equation but also the necessity of evaluating arbitrary constants that make the classical solution laborious. (The number of arbitrary constants is equal to the order of the differential equation.) For these reasons, it is usually expedient to resort to other methods of transient analysis to obtain the pulse-response characteristic of more complex networks.

**13. Equivalent Series Networks.**—There are some additional properties of equivalent series networks that are rather important. One property becomes evident upon examination of some of the equivalent networks in this and succeeding chapters. The number of elements in the equivalent series network is often less than the number of elements in the series-parallel network. This means that the same pulse-response characteristic can be achieved with fewer elements, if other conditions permit. This property can be used to good advantage in some practical cases. However, even though the number of elements is less, the equivalent network may contain a parameter that does not exist in the series-parallel network. For example, some series-parallel  $RC$  networks have been shown to contain inductance in the equivalent series network.

The equivalent series network is capable of giving the same pulse-response characteristic as the actual network that it replaces. However, the total impedance across the output terminals of the equivalent network is generally different from the total impedance across the output terminals of the actual network. This can be an important aspect in some applications. In addition, the values of the equivalent-network parameters are usually dependent upon the parameter that is connected across the output terminals. For instance, the value of the resistance in the equivalent network of Fig. 115 is dependent upon the capacitance  $C$  across the output terminals. Equivalent networks that are obtained by means of Thévenin's theorem have neither of these shortcomings; *i.e.*, the total impedance across the output terminals is the same in both the actual and equivalent networks, and the parameters of the equivalent network do not depend upon the load impedance. However, Thévenin's theorem is usually not well suited to the classical

method of transient analysis except in cases where only one type of parameter is involved.

A review of the method used to obtain equivalent series networks reveals that the equivalent network is valid for any generator voltage whatsoever. This is a significant property. The fact that the equivalent network is not dependent upon the generator voltage can be demonstrated by substituting any generator voltage  $e_g$  for the pulse voltage  $E$  in the differential equation. The equivalent-network parameters are the same, and the equivalent generator voltage will be of the same form as the voltage  $e_g$ . Therefore, if a voltage that is not rectangular is applied to a series-parallel network, the response characteristic can still be found on the basis of an equivalent series network.

**14. Summary.**—The following statements summarize the pertinent features of this chapter:

1. Many series-parallel  $RC$  networks have pulse-response characteristics that are the same as those of series networks.

2. A series network that has the same pulse-response characteristic as a series-parallel network is called an *equivalent series* network.

3. The equivalent series network can be found directly from the differential equation for the series-parallel network, and a solution of the differential equation is not necessary.

4. A series equivalent network does not exist if the differential equation for the series-parallel network is of higher order than the second in  $q$ , or of higher order than the first in  $i$ .

5. When a series equivalent network does not exist, the series-parallel differential equation must be solved completely, and all constants of integration must be evaluated from the initial conditions.

6. The equivalent series network is valid for any generator voltage.

7. The total impedance across the output terminals of the equivalent network is generally different from that of the series-parallel network.

8. When the equivalent network for a series-parallel  $RC$  network contains  $R$ ,  $L$ , and  $C$ , it is invariably overdamped.

9. The procedure for determining the equivalent network is
  - a. Using Kirchhoff's laws, write the instantaneous voltage equations for the series-parallel network.

- b. Decide upon the parameter,  $R$  or  $C$ , that is desired across the output terminals of the equivalent network.
  - c. If  $R$  is chosen as the output parameter, obtain from the voltage equations the differential equation for the instantaneous current through  $R$  as a function of time only. If  $C$  is chosen as the output parameter, solve the voltage equations for the instantaneous charge on  $C$  as a function of time only.
  - d. The coefficients of the differential equation for current, or charge, in terms of time only determine the parameters of the equivalent series network.
10. The value of the equivalent-series-network method lies in the fact that a new network can be regarded as a familiar series network insofar as the pulse-response characteristic is concerned.

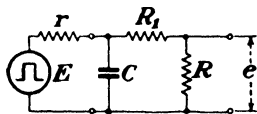
### Problems

**Prob. 1.** The parameters of the network in Fig. 119c have the following values:  $E = 10$  volts,  $T = 0.01$  sec.,  $r = 2,000$  ohms,  $C_1 = 0.25$   $\mu$ f,  $C_2 = 0.4$   $\mu$ f, and  $C = 0.1$   $\mu$ f.

- a. What is the output voltage at  $t = 165$  microseconds?
- b. What are the values of voltage across  $C$ ,  $C_1$ , and  $C_2$  at  $t = 0.01$  sec.?
- c. What is the output voltage at  $t = 0.01132$  sec.?

**Prob. 2.** In the network in Fig. 117,  $r = 1,000$  ohms,  $R = 2,000$  ohms,  $R_1 = 4,000$  ohms,  $C_1 = 0.25$   $\mu$ f,  $C_2 = 0.01$   $\mu$ f, and  $C = 0.04$   $\mu$ f. Show that the network is overdamped for these values of parameters.

**Prob. 3.** Find the equivalent series network for the series-parallel network shown below.



**Prob. 4.** Prove that it is impossible to fulfill conditions for critical damping in the equivalent series network shown in Fig. 123.

**Prob. 5.** What are the equations for  $e_E$  and  $e_0$  in the network in Fig. 125a in terms of the actual network parameters?

## CHAPTER VII

### SERIES-PARALLEL NETWORKS CONTAINING RESISTANCE AND INDUCTANCE

The preceding chapter has shown that equivalent series networks can be found for many series-parallel networks containing resistance and capacitance only. It is also possible to find equivalent series networks for many series-parallel networks containing resistance and inductance only. In this chapter the pulse-response characteristics of series-parallel networks containing resistance and inductance only will be determined by the equivalent-series-network method. The limitations of the method will be pointed out, and a criterion, essentially the same as that for  $RC$  networks, will be developed that will indicate when the method is applicable.

#### RL NETWORKS WITH RESISTANCE ACROSS OUTPUT

The first class of series-parallel networks to be treated is that in which the output pulse appears across a resistance in both the actual and equivalent networks.

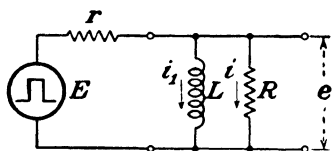


FIG. 126.—Series-parallel  $RL$  network.

**1. Pulse-response Characteristic. Example 1.**—The simple network shown in Fig. 126 will be analyzed first. Recall that the procedure in deriving the equivalent series network is to find the differential equation for the instantaneous current and then to utilize the coefficients of this differential equation to determine the equivalent-series-network parameters. To find the differential equation for  $i$ , the current through the output resistance  $R$ , apply Kirchhoff's laws to the network. Two independent equations for instantaneous voltage during the generator pulse are

$$E = r(i_1 + i) + Ri \quad (132)$$

$$L \frac{di_1}{dt} = Ri \quad (133)$$

Solve Eq. (133) for  $i_1$ .

$$i_1 = \frac{R}{L} \int i dt = \frac{R}{L} q$$

Substitute this value of  $i_1$  into Eq. (132) to obtain the required equation for current through  $R$ .

$$E = r \left( \frac{R}{L} q + i \right) + Ri$$

Collect like terms.

$$E = (R + r)i + \frac{Rr}{L} q$$

The parameters of the equivalent series network are

$$E_s = E$$

$$L_s = 0$$

$$R_s = R + r$$

$$C_s = \frac{L}{Rr}$$

The equivalent network is given in Fig. 127. The pulse-response characteristic of this network, which is the same as that of the network in Fig. 126, has been analyzed in detail in Chap. III.

**2. Comparison of Actual and Equivalent Networks.**—The equivalent network is useful in checking back on the conditions in the actual network both at the time  $t = 0$ , and when steady-state conditions have been reached. At  $t = 0$  in the equivalent network, the output voltage is equal to  $ER/(R + r)$ , which means that the instantaneous value of load current at  $t = 0$  is  $E/(R + r)$ . In the actual network the same is true, because the current through  $L$  is zero at  $t = 0$  and the instantaneous current flowing from the generator is determined by  $(R + r)$ .

When the steady state has been reached, which requires that the generator-pulse width be large compared with the time constant, the equivalent-network current is zero and the output is zero. This is likewise true in the actual network, because the voltage across  $L$ , which is the same as the voltage across  $R$ ,

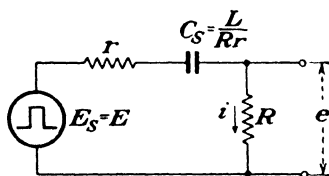


FIG. 127.—Equivalent series network for that in Fig. 126.

must be zero when the steady-state conditions are reached. This is because no voltage appears across  $L$  unless the current is changing.

It is important to realize that these two networks are equivalent only as far as the output pulse is concerned. They are not equivalent as far as the generator is concerned. For instance, the steady-state value of the generator current is zero in the equivalent network while the steady-state value of generator current is  $E/r$  in the actual network.

The network time constant in the equivalent network is  $R_s C_s = (R + r)L/Rr$ . The same time constant results from a combination of  $L$  in series with  $R$  and  $r$  in parallel. This is indicated clearly in Fig. 131, which is another equivalent network for that in Fig. 126.

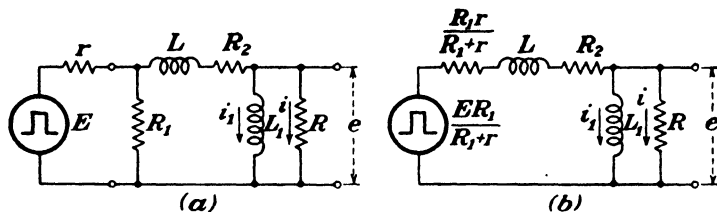


FIG. 128.—Equivalent series-parallel networks as obtained by Thévenin's theorem.

**3. Pulse-response Characteristic. Example 2.**—A more complex series-parallel network containing resistance and inductance only is shown in Fig. 128a. An immediate simplification can be made by the use of Thévenin's theorem:  $R_1$  and  $r$  can be combined and  $E$  replaced by an equivalent generator. The reduced network is given in Fig. 128b. Two independent voltage equations involving  $i_1$  and  $i$  in the reduced network are

$$\frac{ER_1}{R_1 + r} = \frac{R_1 r}{R_1 + r} (i_1 + i) + L \left( \frac{di_1}{dt} + \frac{di}{dt} \right) + R_2 (i_1 + i) + Ri \quad (134)$$

$$L_1 \frac{di_1}{dt} = Ri \quad (135)$$

To find the differential equation for  $i$  in terms of  $t$  only, proceed as follows:

Solve Eq. (135) for  $di_1/dt$ .  $\frac{di_1}{dt} = \frac{R}{L_1} i$

Solve for  $i_1$ .

$$i_1 = \frac{R}{L_1} \int i \, dt = \frac{R}{L_1} q$$

Insert these values of  $i_1$  and  $di_1/dt$  into Eq. (134).

$$\frac{ER_1}{R_1 + r} = \left( \frac{R_1 r}{R_1 + r} + R_2 \right) \left( \frac{R}{L_1} q + i \right) + L \left( \frac{R}{L_1} i + \frac{di}{dt} \right) + Ri$$

Collect like terms.

$$\begin{aligned} \frac{ER_1}{R_1 + r} = L \frac{di}{dt} + \left( \frac{R_1 r}{R_1 + r} + R_2 + \frac{RL}{L_1} + R \right) i \\ + \left( \frac{RR_1 r}{(R_1 + r)L_1} + \frac{R_2 R}{L_1} \right) q \end{aligned}$$

Thus the parameters of the equivalent series network are

$$\begin{aligned} E_s &= \frac{ER_1}{R_1 + r} \\ L_s &= L \\ R_s &= \frac{R_1 r}{R_1 + r} + R_2 + \frac{RL}{L_1} + R \\ C_s &= \frac{L_1}{R \left( \frac{R_1 r}{R_1 + r} + R_2 \right)} \end{aligned}$$

The equivalent series network is shown in Fig. 129. This type of network has been analyzed in Chap. V. In a series

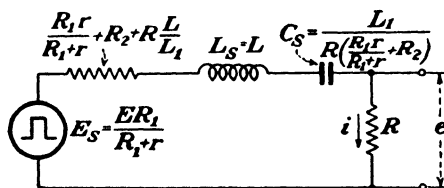


Fig. 129.—Equivalent series network for that in Fig. 128.

network containing resistance, inductance, and capacitance, the relative value of  $R_s$  compared with  $2\sqrt{L_s/C_s}$  determines the form of the output pulse. In this case,  $R_s$  is always greater than  $2\sqrt{L_s/C_s}$ , and consequently the network is overdamped or non-oscillatory. This can be deduced from the actual network where no capacitance exists and where there can be no exchange of

energy in an oscillatory manner. The mathematical proof that this network is overdamped is left to the reader.

**4. Limitations on Equivalent-series-network Method.**—The equivalent-series-network method, which enables a series-parallel network to be regarded as a series network insofar as the pulse-response characteristic is concerned, is applicable to a variety of series-parallel networks. However, an equivalent series network cannot be found for every series-parallel

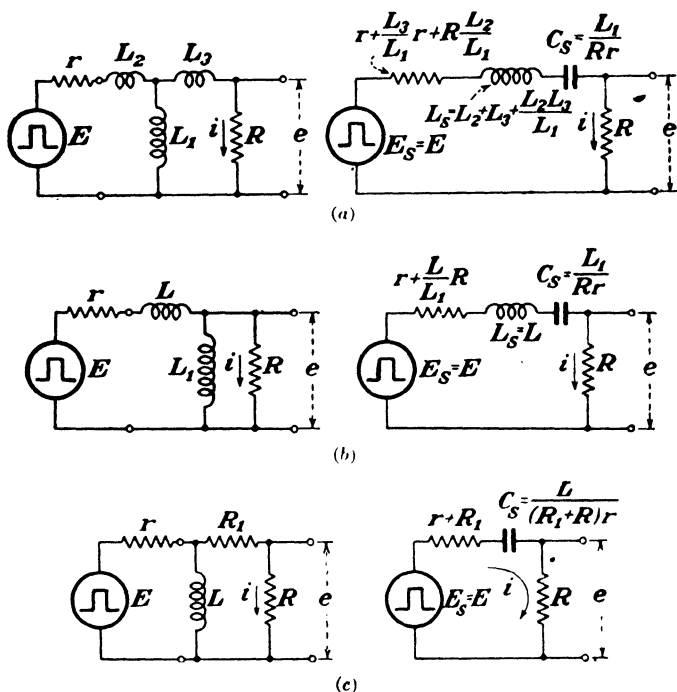


FIG. 130.—Examples of equivalent networks with resistance output.

network that contains inductance and resistance only. The most precise manner in which the limitation can be expressed is in terms of the differential equation that arises from the series-parallel network. If the differential equation for charge is higher than the second order (or if the current equation is higher than the first order), no equivalent series network can represent the pulse-response characteristic. In other words, no linear network parameter placed in a linear series network

gives rises to a differential equation that is of higher order than the second. This is the same fundamental limitation that exists in *RC* networks.

If a third or higher order equation results from a series-parallel network, it is impossible to equate coefficients to those of a series network because terms of higher order than the second are nonexistent in the equation for the series-type network. In this event, the higher order equation must be solved completely, and all arbitrary constants must be evaluated to determine the pulse-response characteristic. Solutions of this type of equation are beyond the scope of this book.

Several series-parallel networks are given in Fig. 130 along with their equivalent series networks. It should be emphasized that the series equivalent network is not unique since it is often possible to find more than one equivalent network. And it should be reiterated that the equivalent series network is exactly equivalent only as far as the output pulse is concerned.

#### ***RL* NETWORKS WITH INDUCTANCE ACROSS OUTPUT**

Another class of series-parallel networks that contains resistance and inductance only is that in which the output pulse appears across an inductance in both the actual and equivalent network. Before analyzing this type of network by the equivalent-series-network method, it is well to examine the limitations of the method in this particular case.

##### **5. Limitation on the Equivalent-series-network Method.—**

The aforementioned limitation on the equivalent-series-network method; namely, the differential equation for charge that arises from the series-parallel network must not be of higher order than the second, is also applicable to equivalent networks where the output pulse appears across an inductance.

There are no other limitations. This, perhaps, may be surprising if it is recalled that in Chap. VI (page 183) an additional limitation was placed upon series equivalent networks that contained a capacitance across the output, especially since Chaps. III and IV showed a very close correspondence between *RL* and *RC* networks. However, a careful examination reveals the difference in the two cases. In both instances the output voltage appears across *L* or across *C*, as the case may be, in both the actual and equivalent networks. In series networks the

steady-state value of voltage during the generator pulse is zero across an inductance and different from zero across a capacitance. Now if the inductance in a series-parallel network is shunted by a resistance, it does not alter the fact that the steady-state value of voltage across the inductance is zero during the generator pulse. This is because the inductance is a short circuit across the resistance when the transient ceases, so no current flows through the resistance in the steady state. It has been pointed out, however, that if a capacitance in a series-parallel network is shunted by a resistance, it is possible in certain networks for the steady-state value of voltage across the capacitance during the generator pulse to be zero. Thus it should be clear that the addition of a resistance in parallel with an inductance cannot cause the steady-state value of voltage across the inductance to differ from the series-network steady-state value; on the other hand, it is quite possible that the addition of a resistance in parallel with a capacitance *can* cause the steady-state value of voltage across the capacitance to differ from the series-network steady-state value. In other words, in *RL* networks of any type the steady-state value of voltage across an inductance is always zero during the generator pulse, while in *RC* networks of any type the steady-state value of voltage across a capacitance depends upon the network configuration.

**6. Pulse-response Characteristic. Example 1.**—To demonstrate that a given series-parallel network can be represented by more than one equivalent series network, the network in Fig. 126 will be analyzed. Figure 127 shows the equivalent series network when the output appears across a resistance. Now it is of interest to find an equivalent series network in which the output appears across the inductance. To do this, the equation for  $i_1$  is required. This equation can be found by utilizing Eqs. (132) and (133). Solve Eq. (133) for  $i$ .

$$i = \frac{L}{R} \frac{di_1}{dt}$$

Substitute this value of  $i$  into Eq. (132).

$$E = r \left( i_1 + \frac{L}{R} \frac{di_1}{dt} \right) + L \frac{di_1}{dt}$$

Collect like terms.

$$E = L \left( 1 + \frac{r}{R} \right) \frac{di_1}{dt} + ri_1$$

Divide both sides of this equation by  $[1 + (r/R)]$ .

$$\frac{ER}{R+r} = L \frac{di_1}{dt} + \frac{Rr}{R+r} i_1$$

The equivalent series network that is represented by this equation is given in Fig. 131. The time constant of this network is exactly the same as the time constant of the network in Fig. 127. The pulse-response characteristic of the network in Fig. 131 was analyzed in detail in Chap. IV. *Figures 131 and 127 are examples of series RL and series RC networks that have exactly the same pulse-response characteristics. The voltage across  $L$  in Fig. 131 and the voltage across  $R$  in Fig. 127 are identical at all times.*

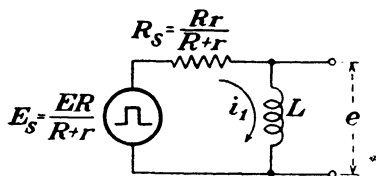


FIG. 131.—Equivalent series network for that in Fig. 126. This network is also equivalent to that in Fig. 127.

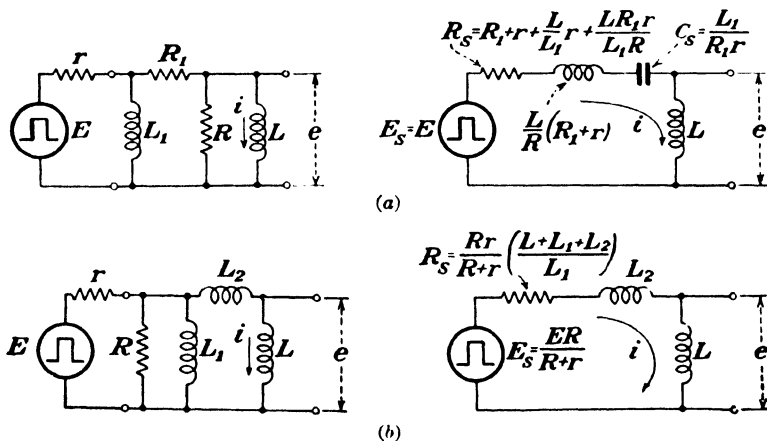


FIG. 132.—Examples of equivalent networks with inductance output.

The manipulation of Eqs. (132) and (133) was hardly necessary in this case because, by Thévenin's theorem, the result

could have been obtained directly from the actual network by combining  $R$  and  $r$  into a single equivalent resistance and replacing the generator by an equivalent series generator.

**7. Pulse-response Characteristic. Other Examples.**—Two

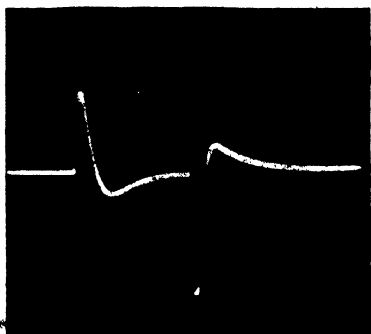


FIG. 133.—Output pulse of the network in Fig. 132*a* when  $T \gg 1/M_s$ . Note the resemblance between this pulse and that in Fig. 101.

additional series-parallel networks containing  $R$  and  $L$  only and having equivalent series networks in which the output appears across  $L$  are shown in Fig. 132. These equivalent networks have been found according to the usual procedure, and no difficulty should be encountered in checking their validity. The oscillogram in Fig. 133 is an illustration of the output voltage obtained for the network in Fig. 132*a* when  $T \gg 1/M_s$ . Evidently the network is over-

damped. This output voltage is

the same as that appearing across  $L$  in an overdamped equivalent series  $RLC$  network.

### A COMPLEX $RL$ NETWORK

The network in Fig. 134*a* is one that cannot be represented by an equivalent series network. Nevertheless, it is informative to analyze it for the following reasons:

1. It is an example that demonstrates how the limitation on the equivalent series method comes about.

2. Under certain special conditions it is possible to find an equivalent series network.

3. The network is exactly the same as many already analyzed if special conditions are imposed. For instance, if  $R_1 = L_1 = \infty$  and  $R_2 = L_2 = 0$ , then the network becomes the same as that in Fig. 126. Another example is when  $L_1 = \infty$ . Then the network becomes the same as that in Fig. 128.

**8. Current Equations.**—Suppose an attempt is made to find the equivalent series network for the network in Fig. 134*a*. If  $L$  is desired for the element across which the output pulse must appear in the equivalent network, the differential equa-

tion for  $i_L$  is required; on the other hand, if  $R$  is to be across the output in the equivalent series network, then the differential equation for  $i_R$  is required. Both differential equations will be found in order to include each of these possibilities.

Before writing any equations the network can be simplified by replacing  $R_1$  and  $r$  by an equivalent series resistance  $R_P$ , equal to  $R_1 r / (R_1 + r)$ , and at the same time replacing  $E$  by an equivalent series generator of voltage equal to  $ER_1 / (R_1 + r)$ . This has been done in Fig. 134*b*. In this network there are three

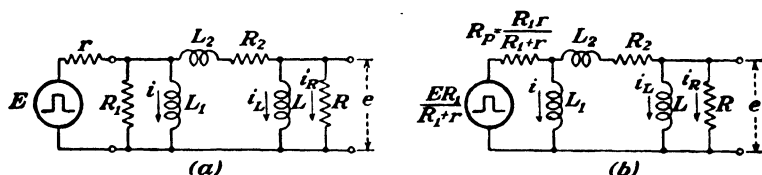


FIG. 134.—Equivalent series-parallel networks as obtained by Thévenin's theorem. This network cannot be represented by an equivalent series network.

independent variables:  $i_L$ ,  $i_R$ , and  $i$ , so three independent voltage equations must be found by Kirchhoff's law. They are

$$\frac{ER_1}{R_1 + r} = R_P(i_L + i_R + i) + L_2 \left( \frac{di_L}{dt} + \frac{di_R}{dt} \right) + R_2(i_L + i_R) + Ri_R \quad (136)$$

$$L \frac{di_L}{dt} = Ri_R \quad (137)$$

$$L_1 \frac{di}{dt} = L_2 \left( \frac{di_L}{dt} + \frac{di_R}{dt} \right) + R_2(i_L + i_R) + Ri_R \quad (138)$$

*Equation for  $i_L$ .*—First the differential equation for  $i_L$  will be found. It is necessary to eliminate  $i_R$  and  $i$  from Eqs. (136), (137), and (138). This can be done by solving Eq. (137) for  $i_R$  and solving Eq. (138) for  $i$ , then substituting these values of  $i_R$  and  $i$  into Eq. (136). The resulting differential equation involving  $i_L$  is

$$\begin{aligned} \frac{ER_1}{R_1 + r} = & \frac{LL_2}{R} \frac{d^2 i_L}{dt^2} + \left( L + L_2 + \frac{LR_2}{R} + \frac{LR_P}{R} + \frac{LL_2 R_P}{L_1 R} \right) \frac{di_L}{dt} \\ & + \left( R_P + R_2 + \frac{LR_P}{L_1} + \frac{L_2 R_P}{L_1} + \frac{LR_2 R_P}{L_1 R} \right) i_L + \frac{R_2 R_P}{L_1} q_L \end{aligned} \quad (139)$$

*Equation for  $i_R$ .*—To find the differential equation for  $i_R$ , Eq. (137) can be utilized.

$$\begin{aligned}\frac{d^2 i_L}{dt^2} &= \frac{R}{L} \frac{di_R}{dt} \\ \frac{di_L}{dt} &= \frac{R}{L} i_R \\ i_L &= \frac{R}{L} q_R \\ q_L &= \frac{R}{L} \int q_R dt\end{aligned}$$

Straightforward substitution into Eq. (139) yields the equation involving  $i_R$ .

$$\begin{aligned}\frac{ER_1}{R_1 + r} &= L_2 \frac{di_R}{dt} + \left( R + \frac{RL_2}{L} + R_2 + R_P + \frac{L_2 R_P}{L_1} \right) i_R \\ &+ \left( \frac{RR_P}{L} + \frac{RR_2}{L} + \frac{RR_P}{L_1} + \frac{RL_2 R_P}{LL_1} + \frac{R_2 R_P}{L_1} \right) q_R \\ &+ \frac{RR_2 R_P}{LL_1} \int q_R dt \quad (140)\end{aligned}$$

Equations (139) and (140) indicate why an equivalent series network is impossible. In Eq. (139) there is a term  $d^2 i_L/dt^2$  which is the same as  $d^2 q_L/dt^2$ , and no constant parameter exists that develops a voltage proportional to the second derivative of the current or third derivative of the charge. Similarly, in Eq. (140) the term  $\int q_R dt$  prohibits a series-network representation because no constant parameter exists that develops a voltage proportional to the time integral of the charge. Aside from these two terms, all others can be represented by equivalent-series parameters.

One might suggest the creation of two fictitious parameters as mathematical devices by which an equivalent network could be concocted. Undoubtedly, this could be done, but recall that the basic advantage of the equivalent-series-network method is that a series-parallel network can be regarded as a series network *whose pulse-response characteristics are already known*. Series networks containing  $R$ ,  $L$ , and  $C$  are the only networks

whose pulse-response characteristics have been analyzed in detail.

An alternative is to consider some special cases of the network in Fig. 134 that will cause the coefficient of the second-order term in Eq. (139) to be zero, or that will cause the coefficient of  $\int q_L dt$  in Eq. (140) to be zero. If this is done, then an equivalent series network is possible.

**9. Equivalent Networks with  $L$  across Output.**—To find an equivalent series network, it is necessary that  $LL_2/R$  be zero; then Eq. (139) will have terms that are in one-to-one correspondence with the general differential equation resulting from a series network. If  $L_2 = 0$ ,  $L = 0$ , or  $R = \infty$ , then  $LL_2/R$  will be zero. The condition that  $L = 0$  is not of interest because this means that the output is short-circuited. Refer to Fig. 134.

If  $L_2 = 0$ , Eq. (139) becomes

$$\begin{aligned} \frac{ER_1}{R_1 + r} = & \left( L + \frac{LR_2}{R} + \frac{LR_P}{R} \right) \frac{di_L}{dt} \\ & + \left( R_P + R_2 + \frac{LR_P}{L_1} + \frac{LR_2R_P}{L_1R} \right) i_L + \frac{R_2R_P}{L_1} q_L \quad (139a) \end{aligned}$$

and the parameters of the equivalent series network are

$$\begin{aligned} E_s &= \frac{ER_1}{R_1 + r} \\ L_s &= L + \frac{L}{R} (R_2 + R_P) \\ R_s &= R_2 + R_P \left( 1 + \frac{L}{L_1} + \frac{R_2L}{RL_1} \right) \\ C_s &= \frac{L_1}{R_2R_P} \end{aligned}$$

This network is shown in Fig. 135a.

If  $R = \infty$ , Eq. (139) becomes

$$\begin{aligned} \frac{ER_1}{R_1 + r} = & (L + L_2) \frac{di_L}{dt} + \left( R_P + R_2 + \frac{LR_P}{L_1} + \frac{L_2R_P}{L_1} \right) i_L \\ & + \frac{R_2R_P}{L_1} q_L \quad (139b) \end{aligned}$$

and the parameters of the equivalent series network are

$$E_s = \frac{ER_1}{R_1 + r}$$

$$L_s = L + L_2$$

$$R_s = R_2 + R_p \left( 1 + \frac{L}{L_1} + \frac{L_2}{L_1} \right)$$

$$C_s = \frac{L_1}{R_2 R_p}$$

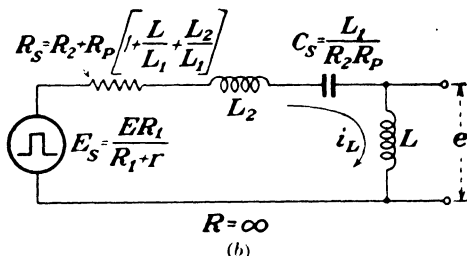
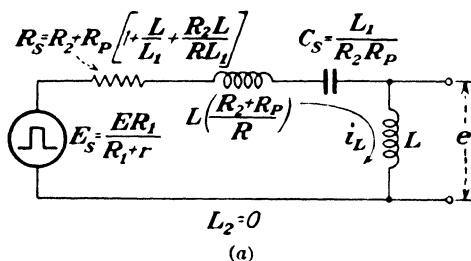


FIG. 135.—Equivalent series networks with inductance output for the network in Fig. 134 under two special conditions.

This network is shown in Fig. 135b.

If additional conditions are imposed upon the equivalent networks, they reduce to those already derived. For instance, if  $R_2 = 0$  and  $R = \infty$  in Fig. 134a, it reduces to the network in Fig. 132b. To demonstrate that the equivalent networks will be the same, refer to Fig. 135b, which is the equivalent network when  $R = \infty$ . When  $R_2 = 0$ , the equivalent capacitance becomes infinite (short circuit) and the equivalent resistance becomes  $R_p \left( 1 + \frac{L}{L_1} + \frac{L_2}{L_1} \right)$ . Meanwhile,  $L$ ,  $L_2$ , and the equivalent generator voltage are unaffected. This results in an equivalent network that is identical to that in Fig. 132b.

**10. Equivalent Networks with  $R$  across Output.**—If the equivalent network is to have  $R$  across the output, then Eq. (140) applies and it is required that  $RR_2R_p/LL_1$  be zero. If  $R = 0$ ,  $R_2 = 0$ ,  $R_1 = 0$ ,  $r = 0$ ,  $L = \infty$ , or  $L_1 = \infty$ , then the coefficient of  $\int q_R dt$  in Eq. (140) will be zero, and there will be a one-to-one correspondence between the terms in Eq. (140) and the terms in the differential equation for a general series network. If  $R = 0$  or  $R_1 = 0$  in the network in Fig. 134, then by inspection the output voltage will always be zero; therefore, these two cases are not of interest.

If  $R_2 = 0$  then Eq. (140) becomes

$$\frac{ER_1}{R_1 + r} = L_2 \frac{di_R}{dt} + \left( R + \frac{RL_2}{L} + R_p + \frac{R_p L_2}{L_1} \right) i_R + \left( \frac{RR_p}{L} + \frac{RR_p}{L_1} + \frac{RL_2 R_p}{LL_1} \right) q_R \quad (140a)$$

and the parameters of the equivalent series network are

$$\begin{aligned} E_s &= \frac{ER_1}{R_1 + r} \\ L_s &= L_2 \\ R_s &= R + R_p + \frac{RL_2}{L} + \frac{R_p L_2}{L_1} \\ C_s &= \frac{L_1 L}{RR_p} \left( \frac{1}{L_1 + \frac{1}{L_2} + L} \right) \end{aligned}$$

If  $r = 0$ , then  $R_p = 0$  and Eq. (140) becomes

$$\frac{ER_1}{R_1 + r} = L_2 \frac{di_R}{dt} + \left( R + \frac{RL_2}{L} + R_2 \right) i_R + \left( \frac{RR_2}{L} \right) q_R \quad (140b)$$

If  $L = \infty$ , Eq. (140) becomes

$$\frac{ER_1}{R_1 + r} = L_2 \frac{di_R}{dt} + \left( R + R_2 + R_p + \frac{L_2 R_p}{L_1} \right) i_R + \left( \frac{RR_p}{L_1} + \frac{R_2 R_p}{L_1} \right) q_R \quad (140c)$$

If  $L_1 = \infty$ , Eq. (140) becomes

$$\frac{ER_1}{R_1 + r} = L_2 \frac{di_R}{dt} + \left( R + \frac{RL_2}{L} + R_2 + R_p \right) i_R + \left( \frac{RR_p}{L} + \frac{RR_2}{L} \right) q_R \quad (140d)$$

The equivalent series networks that are applicable in each of these instances are shown in Fig. 136.

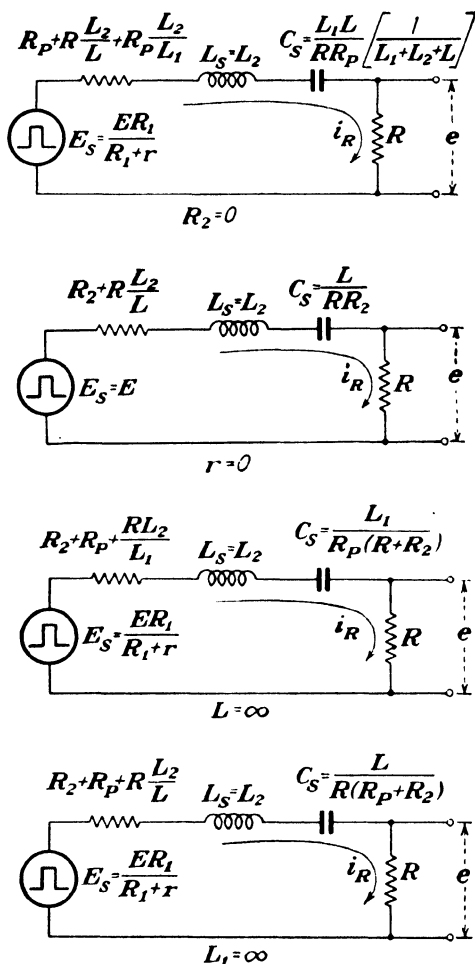


FIG. 136.—Equivalent series networks with resistance output for the network in Fig. 134 under four special conditions.

If additional conditions are imposed upon these equivalent series networks, they reduce to those already derived. In fact, the equivalent network for  $L_1 = \infty$  is the same as that in Fig.

129 because the network in Fig. 134 becomes the same as the network in Fig. 128 when  $L_1 = \infty$ .

### A NETWORK CONTAINING MUTUAL INDUCTANCE

An important type of network that is encountered frequently is one in which there is no direct electrical connection between the output and the input but which transfers voltage through the medium of mutual inductance. The transformer and many radio-frequency networks depend upon this principle. A basic inductively coupled network is given in Fig. 137.

**11. Induced Voltages.**—The network operation depends upon the fact that there is electrical interaction between  $L_1$  and  $L_2$  due to their proximity. This interaction can be explained on the basis of the magnetic field produced by current. Current

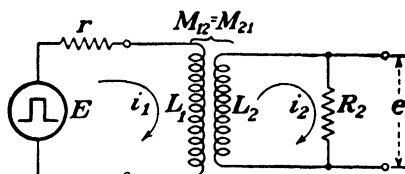


FIG. 137.—A simple inductively coupled network.

is always accompanied by a magnetic field, and when current varies, its accompanying magnetic field varies. A varying magnetic field will cause voltage to be developed in conductors that are close enough to intercept the changing magnetic field. This is called an *induced* voltage. Hence, when  $i_1$  varies in the network shown in Fig. 137, the magnetic field produced by  $L_1$  varies, and this varying field intercepts (or links) the conductors that constitute  $L_2$ . If  $L_1$  and  $L_2$  are in very close proximity, a substantial part of the changing field produced by  $L_1$  will intercept the turns of  $L_2$  and the induced voltage will be large. On the other hand, if  $L_1$  is moved away from  $L_2$ , the intensity of the changing magnetic field that  $L_2$  intercepts is decreased and the induced voltage will decrease.

If the current  $i_1$  is not changing,  $L_1$  produces a constant magnetic field. A constant field is not capable of inducing a voltage in another conductor unless the conductor is in motion. Therefore, if the position of  $L_2$  relative to  $L_1$  is fixed, the voltage

induced in  $L_2$  by the field produced by  $L_1$  is zero when  $i_1$  is constant.

If  $L_2$  is part of a closed network, the induced voltage will cause a flow of current  $i_2$ . This current is also accompanied by a magnetic field. When  $i_2$  is changing,  $L_2$  produces a varying magnetic field that intercepts the turns of  $L_1$  and induces a voltage in  $L_1$ . The induced voltage in  $L_1$  affects the current  $i_1$ . This interaction of  $L_1$  on  $L_2$  and  $L_2$  on  $L_1$  is appropriately described by the term "mutual inductance," since the effects are mutual.

**12. Mutual Inductance.**—The induced voltages depend upon the proximity of the inductors, their physical size and shape, and upon the rate of change of current (or rate of change of magnetic field). To describe these induced voltages in mathematical terms an equation relating induced voltage to rate of change of current is required. Such an equation is

$$e_2 = M_{12} \frac{di_1}{dt} \quad (141)$$

where  $e_2$  is the voltage induced in  $L_2$  due to the varying current  $i_1$  in  $L_1$ , and  $M_{12}$  is the mutual inductance between  $L_1$  and  $L_2$ . When the inductors  $L_1$  and  $L_2$  are in close proximity,  $M_{12}$  is large; when the inductors are moved apart,  $M_{12}$  decreases. A similar equation for the voltage induced in  $L_1$  due to the varying current  $i_2$  can be written.

$$e_1 = M_{21} \frac{di_2}{dt} \quad (142)$$

It will be assumed without proof that the mutual inductance  $M_{21}$  between  $L_2$  and  $L_1$  is the same as that between  $L_1$  and  $L_2$ . It can be shown rigorously that  $M_{12} \equiv M_{21}$ .<sup>1</sup>

Equations (141) and (142) are very similar to the defining equation for inductance [Chap. I, Eq. (4)]. By analogy, then,  $M_{12}$  can be considered to be a network parameter that is an inductance.  $M_{12}$  will be a constant parameter if the magnetic permeability of the medium surrounding  $L_1$  and  $L_2$  is constant. If air surrounds  $L_1$  and  $L_2$ , for example,  $M_{12}$  will be constant, since the magnetic permeability of air is constant for most

<sup>1</sup> R. R. Lawrence, "Principles of Alternating Currents," 2d ed., p. 187, McGraw-Hill Book Company, Inc., New York, 1935.

practical purposes. However, if  $L_1$  and  $L_2$  are wound on an iron core, as is often done to confine most of the magnetic field to a path that will intercept both inductors, then  $M_{12}$  is no longer constant. Since only linear elements are being treated, it is assumed that the magnetic permeability of the medium surrounding  $L_1$  and  $L_2$  is constant.

**13. Differential Equations for Voltage.**—To analyze this magnetically coupled network by the classical method, the differential equation for voltage must be found. Apply Kirchhoff's law to the closed network in which  $i_1$  is flowing.

$$E + e_1 - ri_1 - L_1 \frac{di_1}{dt} = 0 \quad (143)$$

The voltage  $e_1$  is induced in  $L_1$  due to the flow of current  $i_2$  in  $L_2$ . This induced voltage tends to increase  $i_1$  and is therefore of the same polarity as the generator voltage  $E$ . The differential equation for voltage in the closed network in which  $i_2$  is flowing is

$$e_2 - R_2 i_2 - L_2 \frac{di_2}{dt} = 0 \quad (144)$$

where  $e_2$  is the voltage induced in  $L_2$  owing to the flow of current  $i_1$  in  $L_1$ . Since  $e_2$  produces the current  $i_2$ , the polarity of  $e_2$  is opposite to that of the voltage drops across  $R_2$  and  $L_2$ .

Equations (143) and (144) can be rewritten in terms of the defining equations for induced voltage.

$$E = ri_1 + L_1 \frac{di_1}{dt} - M_{21} \frac{di_2}{dt} \quad (143a)$$

$$0 = R_2 i_2 + L_2 \frac{di_2}{dt} - M_{12} \frac{di_1}{dt} \quad (144a)$$

**14. Equivalent Networks.**—A meaningful interpretation of Eqs. (143a) and (144a) has been found in the past by employing a mathematical trick that really depends upon knowing the answer previously. There are two approaches, each of which is essentially the same, that can be employed to find an equivalent network for that in Fig. 137. One is to find some equivalent network that has differential equations identical to Eqs. (143a) and (144a). The other is to manipulate Eqs. (143a) and (144a) until they are in a form that is applicable to an equivalent

network. The equivalent network is shown in Fig. 138 where no mutual inductance exists among the three inductors. The differential equations for this network are the same as Eqs. (143a) and (144a). They are

$$E = ri_1 + (L_1 - M_{21}) \frac{di_1}{dt} + M_{21} \left( \frac{di_1}{dt} - \frac{di_2}{dt} \right) \quad (143b)$$

$$0 = R_2 i_2 + (L_2 - M_{12}) \frac{di_2}{dt} + M_{12} \left( \frac{di_2}{dt} - \frac{di_1}{dt} \right) \quad (144b)$$

The fact that these equations are the same as Eqs. (143a) and (144a) can be verified by expanding them. Upon expanding

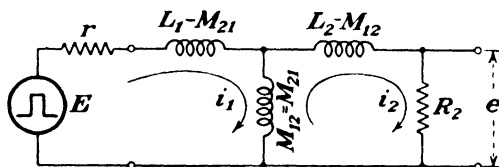


FIG. 138.—Equivalent network for that in Fig. 137, which is valid for both the generator and the load. Both  $i_1$  and  $i_2$  flow through the mutual inductance.

Eq. (143b), the term  $\left( -M_{21} \frac{di_1}{dt} + M_{21} \frac{di_1}{dt} \right)$  drops out; in Eq. (144b) the term  $\left( -M_{12} \frac{di_2}{dt} + M_{12} \frac{di_2}{dt} \right)$  drops out. In each case the remaining terms are the same as those in Eqs. (143a) and (144a).

According to the second approach, the term  $M_{21}(di_1/dt)$  is added to and subtracted from Eq. (143a), and the term  $M_{12}(di_2/dt)$  is added to and subtracted from Eq. (144a). Then the terms are grouped as shown in Eqs. (143b) and (144b) and the equivalent network in Fig. 138 is deduced.

This equivalent network shows very clearly that the role of mutual inductance can be considered to be the same as that played by an ordinary inductance. The networks in Fig. 137 and 138 are equivalent as far as both the load and the generator are concerned.

Now that an equivalent network has been found that represents the mutual inductance as an ordinary inductance, it is possible to find the pulse-response characteristics according to previously developed methods. Specifically, it is possible

to find an equivalent series network for this series-parallel network, and thereby to determine the pulse-response characteristics in terms of a known network. The network in Fig. 130a is the same as that in Fig. 138 except for the values of the parameters, and therefore the equivalent series network has already been determined. It is simply a matter of replacing  $L_1$ ,  $L_2$ , and  $L_3$  in Fig. 130a by  $M_{12}$ ,  $(L_1 - M_{21})$ , and  $(L_2 - M_{12})$ , respectively. The equivalent series network and, of course, the actual network in Fig. 137 are invariably overdamped because no capacitance exists in the actual network.

### CONCLUSION

Many of the concluding remarks at the end of Chap. VI are also appropriate for series-parallel *RL* networks. At the risk of unnecessary repetition, some of them are restated here for emphasis and to summarize the high lights of this chapter.

1. Numerous series-parallel *RL* networks, including magnetically coupled networks, have pulse-response characteristics that are the same as those of series networks.

2. The equivalent series network that has the same pulse-response characteristic as a series-parallel network can be found directly from the differential equations, and a solution of the differential equations is not necessary.

3. No equivalent series network exists if the differential equation for the series-parallel network is of higher order than the second in  $q$  or of higher order than the first in  $i$ .

4. The higher order differential equation must be solved completely when no equivalent series network exists, and constants of integration, equal in number to the order of the differential equation, must be evaluated from the initial conditions.

5. The equivalent series network is valid for any generator voltage.

6. The total impedance across the output terminals of the equivalent network is generally different from that of the series-parallel network. In addition, the equivalent series network is not equivalent to the series-parallel network insofar as the generator is concerned.

7. Series-parallel *RL* networks are invariably overdamped, and when they give rise to equivalent *RLC* networks, the equivalent networks are likewise overdamped.

8. The procedure for determining the equivalent network is as follows:

- a. Write the instantaneous-voltage equations for the series-parallel network by utilizing Kirchhoff's laws.
- b. Decide upon the parameter,  $R$  or  $L$ , that is desired across the output terminals of the equivalent network.
- c. If  $R$  is chosen as the output parameter, obtain from the voltage equations the differential equation for the instantaneous current through  $R$  as a function of time only. If  $L$  is chosen as the output parameter, solve the voltage equations for the instantaneous current through  $L$  as a function of time only.
- d. The coefficients of this differential equation for current determine the parameters of the equivalent series network.

9. In the case of magnetically coupled networks, the equivalent network that represents the mutual inductance as an ordinary inductance parameter must be found first. Thereafter, the procedure in obtaining the equivalent series network is the same as that used in ordinary  $RL$  networks.

10. The value of the equivalent-series-network method lies in the fact that a new network can be regarded as a familiar series network insofar as the pulse-response characteristic is concerned.

### Problems

**Prob. 1.** The parameters of the network in Fig. 130c have the following values:  $E = 80$  volts,  $T = 350$  microseconds,  $R = 10,000$  ohms,  $R_1 = 2,000$  ohms, and  $r = 8,000$  ohms.

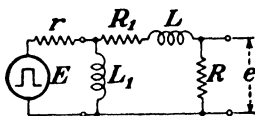
- a. What is the time constant of the network?
- b. At what instant during the generator pulse is the output voltage equal to 24 volts?
- c. What current flows through  $L$  at  $t = 0$ ? At  $t = T$ ?
- d. What is the most negative value of output voltage?

**Prob. 2.** In the network in Fig. 126 the positive output voltage at  $t = T$  is equal in magnitude to the maximum negative output voltage. If  $T = 208$  microseconds, what is the time constant of the network?

**Prob. 3.** In the networks in Fig. 130b,  $R = r = 4,000$  ohms,  $L_1 = L$ , and  $M_s = 6 \times 10^4$  sec.<sup>-1</sup>

- a. At what instant during the generator pulse will the output voltage be a maximum?
- b. What is the value of  $L$ ?

**Prob. 4.** Find the equivalent series network for the series-parallel network shown below.



**Prob. 5.** In the network in Fig. 137,  $E = 20$  volts,  $T = 50$  microseconds,  $L_1 = 0.004$  henry,  $L_2 = 0.016$  henry,  $M_{12} = M_{21} = 0.002$  henry, and  $R_2 = 0$ .

- a. What is the voltage across  $r$  at  $t = 1.5$  microseconds?
- b. What is the maximum current through  $L_2$ ? (Assume that all inductors have negligible resistance.)

## CHAPTER VIII

### SERIES-PARALLEL NETWORKS CONTAINING RESISTANCE, INDUCTANCE, AND CAPACITANCE

Series-parallel networks containing resistance and either capacitance or inductance have been discussed in Chaps. VI and VII. In many instances, series-parallel networks containing resistance, inductance, *and* capacitance are used. It is therefore of interest to analyze such networks.

The scope of the equivalent-series-network method of analysis is generally more confined in the case of series-parallel *RLC* networks than in the case of series-parallel *RL* or *RC* networks. This should not be surprising because many series-parallel *RL* or *RC* networks give rise to equivalent series networks that contain *R*, *L*, and *C*. When the series-parallel network *itself* contains *R*, *L*, and *C*, it is less likely that an equivalent series network exists; *i.e.*, series-parallel combinations of *R*, *L*, and *C* frequently behave in a manner that is too complicated to describe by the most general series-network differential equation.

Equivalent series networks can be found whenever the differential equation for charge resulting from the series-parallel *RLC* network is no higher than the second order. This same condition limits the applicability of the method in all types of networks, and it is a fundamental limitation.

#### *RLC* NETWORKS WITH CAPACITANCE ACROSS OUTPUT

Networks that have a capacitance across the output occur very often in practice. The capacitance output can be an actual capacitor connected across the output or a stray capacitance that is an inherent part of the network. Stray capacitances are discussed in Chap. IX.

**1. Pulse-response Characteristic. Example 1.**—Figure 139 shows a network with capacitance output that can be analyzed by the equivalent-series-network method. The equation for charge on *C* is required. To find this equation Kirchhoff's laws

can be applied to the network. Two independent voltage equations are

$$E = r(i_1 + i) + L\left(\frac{di_1}{dt} + \frac{di}{dt}\right) + \frac{q}{C} \quad (145)$$

$$Ri_1 = \frac{q}{C} \quad (146)$$

To eliminate  $i_1$  from these equations, solve Eq. (146) for  $i_1$ , and substitute this value into Eq. (145).

$$i_1 = \frac{q}{RC}$$

$$\frac{di_1}{dt} = \frac{i}{RC}$$

$$E = r\left(\frac{q}{RC} + i\right) + L\left(\frac{i}{RC} + \frac{di}{dt}\right) + \frac{q}{C}$$

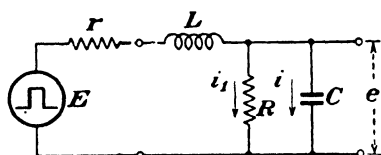


FIG. 139.—Series-parallel RLC network with a rectangular-pulse generator.

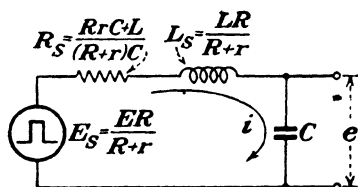


FIG. 140.—Equivalent series network for that in Fig. 139.

Collect like terms.

$$E = L \frac{d^2q}{dt^2} + \left(r + \frac{L}{RC}\right) \frac{dq}{dt} + \frac{(R+r)}{R} \frac{q}{C}$$

Multiply both sides of the equation by  $R/(R+r)$ .

$$\frac{ER}{R+r} = \frac{LR}{R+r} \frac{d^2q}{dt^2} + \left[\frac{RrC + L}{(R+r)C}\right] \frac{dq}{dt} + \frac{q}{C}$$

The parameters of the equivalent series network, as determined by the coefficients of this equation for  $q$ , are

$$E_s = \frac{ER}{R+r}$$

$$R_s = \frac{RrC + L}{(R+r)C}$$

$$L_s = \frac{LR}{R+r}$$

$$C_s = C$$

The equivalent network is given in Fig. 140. The behavior of

charge on  $C$  in both the actual and equivalent networks is identical and therefore the pulse-response characteristics of the two networks are identical.

In the equivalent network the steady-state value of output voltage during the generator pulse is equal to the generator voltage  $ER/(R + r)$ , because no current flows when the steady state is reached. In the actual network the steady-state value of the output voltage during the generator pulse is also equal to  $ER/(R + r)$ , but in this case the current in the network is  $E/(R + r)$ .

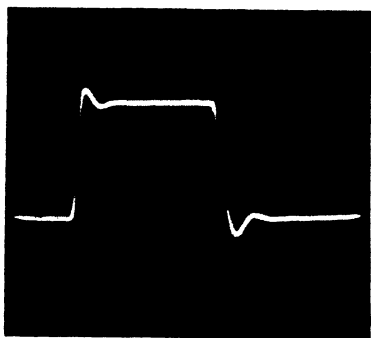


FIG. 141.—Output pulse of the network in Fig. 139 for the oscillatory case when  $R_s$  is only slightly less than  $2\sqrt{L_s/C}$  and  $T > 1/M_s$ .

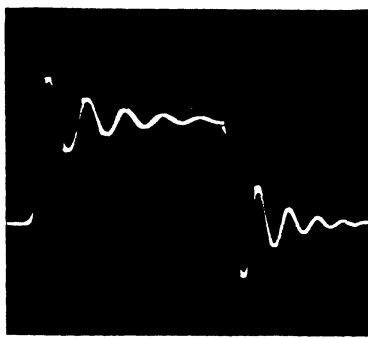


FIG. 142.—Output pulse of the network in Fig. 139 for the oscillatory case when  $R_s$  is much less than  $2\sqrt{L_s/C}$ . The transient is almost negligible at  $t = T$ .

The pulse-response characteristics of series  $RLC$  networks have been determined in Chap. V. Recall that the form of the output pulse is dependent upon the relative values of the network parameters. Specifically, if  $R_s > 2\sqrt{L_s/C}$ , the network is overdamped or nonoscillatory; if  $R_s < 2\sqrt{L_s/C}$ , the network is oscillatory; and if  $R_s = 2\sqrt{L_s/C}$ , the network is critically damped and just on the verge of oscillation. In Chaps. VI and VII, equivalent series networks containing  $R$ ,  $L$ , and  $C$  were invariably overdamped, because an oscillatory exchange of energy could not take place in the actual network in the absence of either  $L$  or  $C$ . In this instance, however, the actual network contains both  $L$  and  $C$ , so under certain conditions it is possible for oscillation to take place. The oscillograms in Figs. 141 and 142 illustrate two possible forms of oscillatory output

voltage for the network in Fig. 139. In Fig. 141 the transient is completely negligible at  $t = T$ , while in Fig. 142 the transient is very small at  $t = T$ . Each of these output voltages can be obtained across  $C$  in a series  $RLC$  network, which is additional evidence that the pulse-response characteristic of the series-parallel network in Fig. 139 is the same as that of the series network in Fig. 140.

An investigation of the relative values of  $R_s$ ,  $L_s$ , and  $C$  in the equivalent network leads to the conditions on the actual network parameters for three possible cases: overdamped, oscillatory, and critically damped.

*Case 1. Overdamped.*—In the overdamped case  $R_s > 2 \sqrt{L_s/C}$  or in terms of the actual network parameters

$$\frac{RrC + L}{(R + r)C} > 2 \sqrt{\frac{LR}{(R + r)C}}$$

A numerical example will serve to show that this inequality is possible. Suppose that  $L = 0.25$  henry,  $C = 0.25 \mu f$ ,  $R = 1,000$  ohms, and  $r = 10,000$  ohms. Substitute these specific values into the above inequality.

$$\begin{aligned} \frac{1,000 \times 10,000 \times 0.25 \times 10^{-6} + 0.25}{(1,000 + 10,000)0.25 \times 10^{-6}} \\ &> 2 \sqrt{\frac{0.25 \times 1,000}{(1,000 + 10,000)0.25 \times 10^{-6}}} \\ 1,000 &> \frac{2,000}{\sqrt{11}} \end{aligned}$$

*Case 2. Oscillatory.*—In the oscillatory case  $R_s < 2 \sqrt{L_s/C}$  or  $\frac{RrC + L}{(R + r)C} < 2 \sqrt{\frac{LR}{(R + r)C}}$ . To demonstrate that oscillation is possible, suppose that the values of the parameters are  $L = 0.25$  henry,  $C = 0.25 \mu f$ ,  $R = 10,000$  ohms, and  $r = 1,000$  ohms. Substitute these values into the above inequality.

$$\begin{aligned} \frac{10,000 \times 1,000 \times 0.25 \times 10^{-6} + 0.25}{(10,000 + 1,000)0.25 \times 10^{-6}} \\ &< 2 \sqrt{\frac{0.25 \times 10,000}{(10,000 + 1,000)0.25 \times 10^{-6}}} \\ 1,000 &< 2,000 \sqrt{10/11} \end{aligned}$$

Case 3. *Critically Damped*.—For critical damping

$$R_s = 2 \sqrt{\frac{Ls}{C}}$$

or  $\frac{RrC + L}{(R + r)C} = 2 \sqrt{\frac{LR}{(R + r)C}}$ . Critical damping is possible since both the overdamped and oscillatory cases can exist. The actual values of the parameters can be found from the above equation for critical damping. One set of values that will satisfy this equation is  $L = 0.364$  henry,  $C = 0.25 \mu\text{f}$ ,  $R = 500$  ohms, and  $r = 500$  ohms.

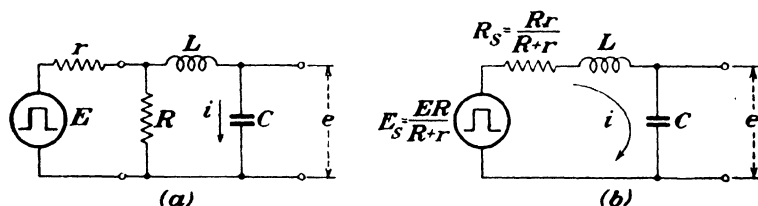


FIG. 143.—Equivalent  $RLC$  networks with capacitance output that can be obtained by means of Thévenin's theorem.

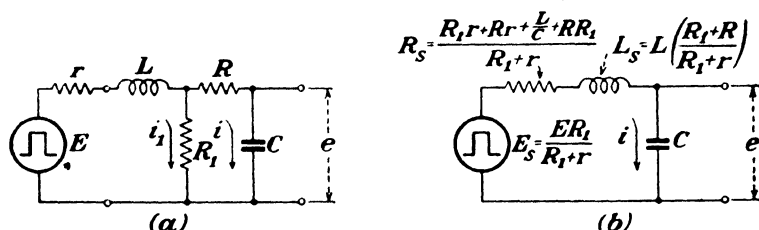


FIG. 144.—Equivalent  $RLC$  networks with capacitance output.

**2. Other Examples.**—The networks given in Figs. 143 and 144 are other examples of series-parallel  $RLC$  networks with a capacitance across the output that can be regarded as series networks insofar as the output pulse is concerned.

Whenever equivalent networks are used, it is well to check at least the steady-state value of output voltage during the generator pulse. The steady-state output voltage after the generator pulse is, of course, invariably zero. The equivalent network for that shown in Fig. 143a can be obtained solely by use of Thévenin's theorem. In Fig. 143b the steady-state output voltage during the generator pulse is equal to the gen-

erator voltage  $ER/(R + r)$ , and the equivalent generator current is zero. In Fig. 143a the steady-state voltage also equals  $ER/(R + r)$ , but in this case the generator current equals  $E/(R + r)$ . This current flows through  $R$  and  $r$  only and maintains the charge on  $C$  at a value  $CER/(R + r)$ .

The equivalent network in Fig. 144b can be found from the actual network in Fig. 144a by the usual method. In the actual network the steady-state value of output voltage during the generator pulse is reached when the currents are constant; i.e.,  $i = 0$  and  $i_1 = E/(R_1 + r)$ . Therefore, the steady-state output voltage is  $ER_1/(R_1 + r)$ . This same output voltage exists in the equivalent network when the steady state is reached; however, in this case the generator current is zero.

In general, series-parallel RLC networks can be either overdamped, oscillatory, or critically damped. This can be verified most conveniently by considering extreme cases in the actual network and examining the relative values of  $R_s$  and  $2\sqrt{L_s/C_s}$ . The network condition for specific parameters can be deduced by evaluating  $R_s$  and  $2\sqrt{L_s/C_s}$ .

### RLC NETWORKS WITH RESISTANCE ACROSS OUTPUT

Two networks that have a resistance across the output are shown in Figs. 145 and 146 along with their equivalent series

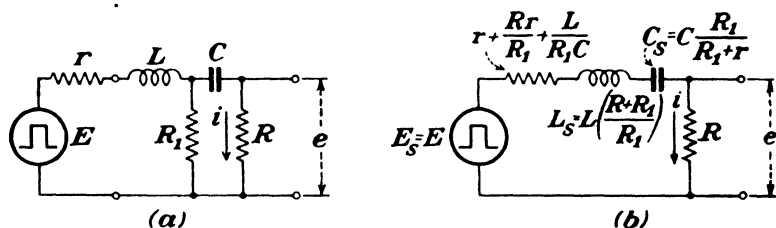


FIG. 145.—Equivalent RLC networks with resistance output.

networks. As an illustration of how these equivalent networks are obtained, the equivalent series network given in Fig. 146b will be derived. Kirchhoff's law yields two independent voltage equations involving the branch currents  $i_1$  and  $i$ .

$$E = r(i_1 + i) + \frac{q}{C} + \frac{q_1}{C} + L \frac{di}{dt} + Ri \quad (147)$$

$$R_1 i_1 = L \frac{di_1}{dt} + Ri \quad (148)$$

To eliminate  $i_1$  from these equations and thereby to obtain the differential equation for  $i$ , solve Eq. (148) for  $i_1$ .

$$i_1 = \frac{R}{R_1} i + \frac{L}{R_1} \frac{di}{dt}$$

Since  $q_1 = \int i_1 dt$ , then

$$q_1 = \frac{R}{R_1} q + \frac{L}{R_1} i$$

Substitute  $q_1$  and  $i_1$  into Eq. (147) and group terms.

$$E = L \left( \frac{R_1 + r}{R_1} \right) \frac{di}{dt} + \left( R + r + \frac{Rr}{R_1} + \frac{L}{R_1 C} \right) i + \left( \frac{R_1 + R}{R_1 C} \right) q$$

The coefficients of this equation determine the equivalent-series-network parameters that are given in Fig. 146b.

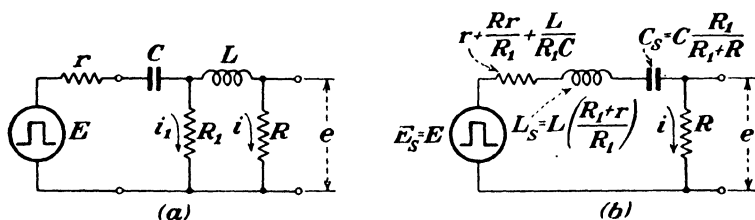


Fig. 146. —Equivalent  $RLC$  networks with resistance output.

Rather than show mathematically how this network can be oscillatory or nonoscillatory from the value of  $R_s$  compared with  $2\sqrt{L_s/C_s}$ , it is informative to investigate the physical considerations in the actual network. By deduction it can be seen from Fig. 146a that oscillation is possible if  $R_1$  is made increasingly large while  $(R + r)$  is made diminishingly small. The argument is that if  $R_1$  is large enough, it can be removed from the network without appreciable effect, and then if

$$(R + r) < 2\sqrt{\frac{L}{C}}$$

oscillation will take place. On the other hand, even if  $R_1$  is removed from the network, oscillation will not take place unless  $(R + r)$  is less than  $2\sqrt{L/C}$ . Since both the oscillatory and overdamped cases are possible, it follows that the critically damped case can exist.

The oscillogram in Fig. 147 demonstrates an oscillatory output voltage obtained for the network in Fig. 146. The oscillogram in Fig. 148 is another illustration of oscillatory output voltage when the values of the network parameters are changed

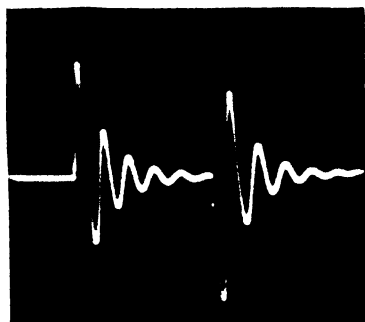


FIG. 147.—Oscillatory output voltage of the network in Fig. 146 when the generator-pulse width is long enough to allow the transient to become small at  $t = T$ .

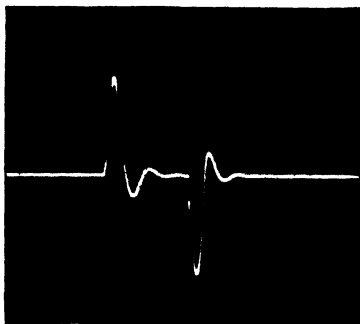


FIG. 148.—Oscillatory output voltage of the network in Fig. 146 when  $R_S$  is only slightly less than  $2\sqrt{L_S/C_S}$  and  $T \gg 1/M_S$ .

slightly. The close similarity between Fig. 147 and Fig. 90 in Chap. V indicates that this pulse-response characteristic is essentially the same as that of a series  $RLC$  network. The sharp discontinuity and bright spot in both Figs. 146 and 147 at  $t = T$  is due to the distributed capacitance of the inductor that was neglected in the analysis.

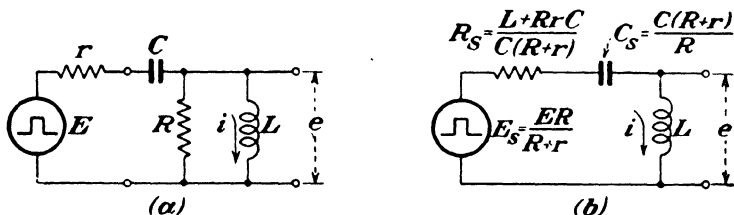


FIG. 149.—Equivalent  $RLC$  networks with inductance output.

#### RLC NETWORKS WITH INDUCTANCE ACROSS OUTPUT

Three series-parallel  $RLC$  networks are shown in Figs. 149, 150, and 151 along with equivalent series networks that have the same pulse-response characteristics. The equivalent networks have been derived in the usual manner. In all of these

networks, both actual and equivalent, the steady-state value of output voltage is zero at all times. Another similarity among

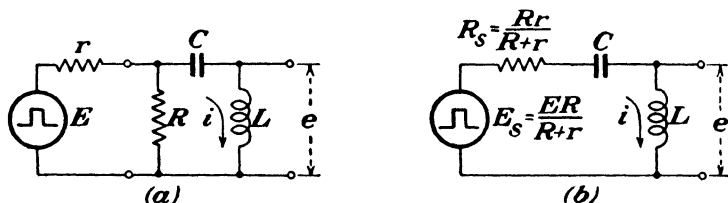


FIG. 150.—Equivalent  $RLC$  networks that can be verified by Thévenin's theorem.

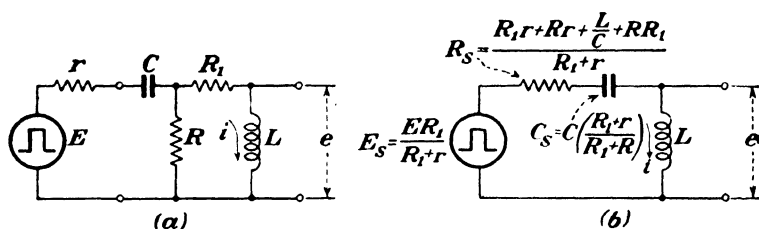


FIG. 151.—Equivalent  $RLC$  networks with inductance output.

the three networks is that an increase in  $R$  tends to encourage oscillation while a decrease in  $R$  tends to bring about an over-damped condition.

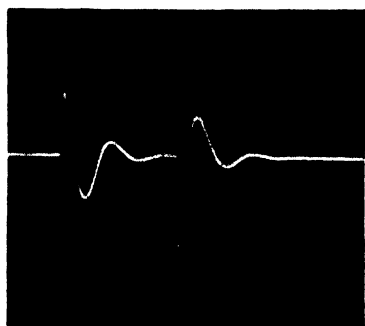


FIG. 152.—Output voltage of the network in Fig. 149a when  $R_S$  is only slightly less than  $2\sqrt{L_S/C_S}$ . The transient is practically negligible at  $t = T$ .

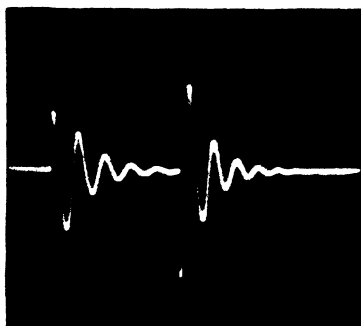


FIG. 153.—Oscillatory output voltage of the network in Fig. 151a when  $R_S$  is much less than  $2\sqrt{L_S/C_S}$ .

The oscillogram in Fig. 152 is the output voltage of the network in Fig. 149a for the oscillatory case. This output

voltage is practically the same as that in Fig. 102, Chap. V, and emphasizes the fact that this series-parallel network has a pulse-response characteristic that is the same as that of an equivalent series network. The oscillogram in Fig. 153 is an oscillatory output voltage obtained from the network in Fig. 151. In both of these oscillograms, the maximum negative value of output voltage exceeds the maximum positive output voltage because the generator pulse was not perfectly rectangular; *i.e.*, the leading edge of the generator pulse at  $t = 0$  was not as steep as the trailing edge at  $t = T$ .

### A NETWORK CONTAINING $L$ AND $C$ IN PARALLEL

An examination of all of the networks in this chapter reveals that none contains an inductor that is connected directly in parallel with a capacitor. Nevertheless, networks that contain  $L$  and  $C$  in parallel are frequently encountered, and their pulse-response characteristics are often desired.

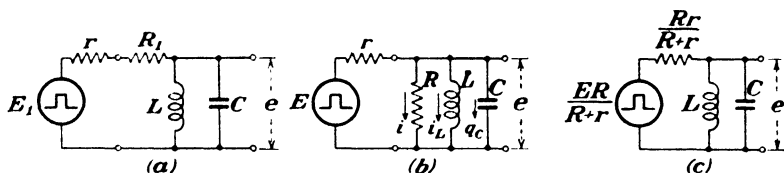


FIG. 154.—Networks containing  $L$  and  $C$  connected directly in parallel.

The parallel  $LC$  networks shown in Fig. 154 are simple examples that will serve to demonstrate the method of solution. Figures 154*b* and 154*c* indicate that a resistance in parallel with  $L$  and  $C$  can be replaced by an equivalent series resistance if the generator voltage is appropriately altered. Therefore, an analysis based upon the network in Fig. 154*b* will include the series resistance case of Fig. 154*a*.

**3. Selection of Output Parameter.**—The pulse-response characteristics of the network in Fig. 154*b* can be found by the equivalent-series-network method. Since  $R$ ,  $L$ , and  $C$  are each connected across the output, there are three possible choices of output parameter, but an equivalent series network is not possible unless the output parameter is chosen as  $R$ . This fact can be deduced by trial from the differential equations. However, it is not necessary to manipulate the equations to realize that a resistance output in the equivalent series network is

the only one that will satisfy some of the obvious properties of this series-parallel network. The reasoning is as follows:

1. The equivalent series network certainly must contain  $R$ ,  $L$ , and  $C$  because oscillation is possible in the series-parallel network, for instance, when  $R$  is very large and  $r$  is small compared with  $2\sqrt{L/C}$ .

2. The steady-state value of output voltage during the generator pulse is zero because the voltage across  $L$  is zero when the current through  $L$  is constant. This current is  $E/r$  when the steady state is reached.

3. The output voltage at the instant the generator pulse appears must be zero because a finite time is required for charge to flow onto  $C$ .

On the basis of these three observations, examine a general series network that contains  $R$ ,  $L$ , and  $C$ . If  $C$  is connected across the output, then the steady-state value of output voltage during the generator pulse is  $E$  and is contrary to point 2. If  $L$  is connected across the output, the output voltage at the instant the generator pulse arrives is  $E$ , which is contrary to point 3. If  $R$  is connected across the output, however, the steady-state value of output voltage during the generator pulse is zero, as is the output voltage at the instant the generator pulse arrives. While this reasoning does not provide sufficient evidence to conclude that an equivalent network is possible with resistance output, it shows, at least, that one containing  $L$  or  $C$  across the output is impossible.

**4. Equivalent Series Network.**—To find the equivalent series network that contains  $R$  across the output, it is necessary to obtain the differential equation for  $i$ , the current through  $R$ , in terms of time only. When Kirchhoff's laws are applied to the network in Fig. 154b, three independent voltage equations result:

$$E = r(i + i_L + i_c) + Ri \quad (149)$$

$$L \frac{di_L}{dt} = Ri \quad (150)$$

$$\frac{q_c}{C} = Ri \quad (151)$$

To eliminate  $i_L$  and  $i_c$  from Eq. (149) proceed as follows: Rewrite Eq. (150).

$$di_L = \frac{R}{L} i dt$$

and integrate.

$$i_L = \frac{R}{L} \int i \, dt = \frac{R}{L} q$$

Rewrite Eq. (151),

$$q_c = RCi$$

and differentiate.

$$\frac{dq_c}{dt} = i_c = RC \frac{di}{dt}$$

Substitute these values of  $i_L$  and  $i_c$  into Eq. (149).

$$E = r \left( i + \frac{R}{L} q + RC \frac{di}{dt} \right) + Ri$$

When like terms in this equation are grouped, the desired form of the equation for  $i$  in terms of time only results.

$$E = RrC \frac{di}{dt} + (R + r)i + \frac{Rr}{L} q \quad (152)$$

Equation (152) determines the parameters of the equivalent series network that is illustrated in Fig. 155. The pulse-response characteristics of this network have been treated in detail in Chap. V.

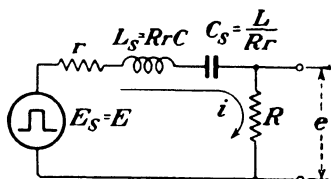


FIG. 155.—Equivalent series network for that in Fig. 154b.

### A SPECIAL NETWORK

The network shown in Fig. 156 is exceptionally interesting,

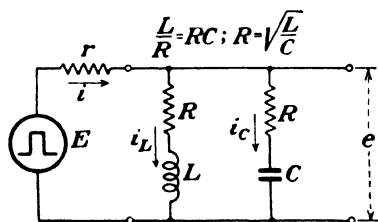


FIG. 156.—A special series-parallel RLC network.

although it is very cumbersome to analyze by the classical method for general values of resistance, inductance, and capacitance. However, when  $R$ ,  $L$ , and  $C$  have the special values indicated in Fig. 156 (equal time constants and equal resistances in the two series branches), it is possible to learn something

about the pulse-response characteristic without excessive labor.

The striking property of this network with the assigned values of  $R$ ,  $L$ , and  $C$  is indicated by Fig. 157, which is an equiv-

alent series network. Insofar as the pulse-response characteristic is concerned, the networks in Figs. 156 and 157 are exactly the same. This can be deduced quite readily by observing that the voltage across  $r$  is the same in both networks if  $i$  is the same in both networks. The output voltage is  $E - ri$ .

**5. Proof of Equivalence.**—The usual method of finding the equivalent series network is not applicable in this case because neither  $i_L$  nor  $i_c$  appears separately in the equivalent network. A special procedure, which follows, is required to prove that the network in Fig. 157 is equivalent to that in Fig. 156.

Three independent voltage equations that apply during the generator pulse are

$$e = Ri_L + L \frac{di_L}{dt} \quad (153)$$

$$e = Ri_c + \frac{qc}{C} \quad (154)$$

$$e = E - ri \quad (155)$$

Equation (153) can be written

$$\frac{e dt}{C} = \frac{Ri_L dt}{C} + \frac{L}{C} di_L$$

and then integrated.

$$\frac{1}{C} \int e dt = \frac{R}{C} q_L + \frac{L}{C} i_L$$

Since  $L/C = R^2$ , this equation can be written

$$\frac{1}{RC} \int e dt = \frac{q_L}{C} + Ri_L \quad (156)$$

Differentiate Eq. (154) and multiply by  $L$ .

$$L \frac{de}{dt} = LR \frac{di_c}{dt} + \frac{L}{C} i_c$$

When  $L/C$  is replaced by its equal  $R^2$ , this equation becomes

$$\frac{L}{R} \frac{de}{dt} = L \frac{di_c}{dt} + Ri_c \quad (157)$$

Adding  $e$  to the left side of Eqs. (156) and (157), and

$$\left( Ri_L + L \frac{di_L}{dt} \right) \quad \text{and} \quad \left( Ri_c + \frac{qc}{C} \right)$$

to the right side of Eqs. (156) and (157), respectively, does not invalidate the equations.

$$e + \frac{1}{RC} \int e \, dt = 2Ri_L + \frac{q_L}{C} + L \frac{di_L}{dt} \quad (156a)$$

$$e + \frac{L}{R} \frac{de}{dt} = 2Ri_c + \frac{q_c}{C} + L \frac{di_c}{dt} \quad (157a)$$

Add Eqs. (156a) and (157a).

$$2e + \frac{1}{RC} \int e \, dt + \frac{L}{R} \frac{de}{dt} = 2R(i_L + i_c) + \frac{q_L + q_c}{C} + L \left( \frac{di_L}{dt} + \frac{di_c}{dt} \right)$$

However,  $i_L + i_c = i$ .

$$2e + \frac{1}{RC} \int e \, dt + \frac{L}{R} \frac{de}{dt} = 2(Ri) + \frac{1}{RC} \int (Ri) \, dt + \frac{L}{R} \frac{d(Ri)}{dt} \quad (158)$$

Equation (158) is true only if  $e = Ri$ , or in other words if

$$i = \frac{e}{R} = \frac{E - ri}{R}$$

Solve for  $i$ .

$$i = \frac{E}{R + r}$$

Therefore, the output voltage becomes

$$e = Ri = \frac{ER}{R + r}$$

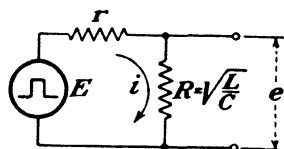


FIG. 157.—Equivalent network for that in Fig. 156 provided  $R = R$  and  $L/R = RC$ .

and the validity of the equivalent network in Fig. 157 is proved.

**6. Network Behavior.**—A physical explanation of the equivalent network is easily made when the behavior of  $i_L$  and  $i_c$  is investigated. Figure 158 shows  $i_L$  and  $i_c$  and their sum  $i$ . For the condition  $R = \sqrt{L/C}$ , the sum of  $i_L$  and  $i_c$  is seen to be constant. Thus, as far as the generator is concerned, the current is flowing just as though a resistor were connected to the generator terminals.

The voltages across the individual parameters of the network bear certain relationships to each other. For instance, the voltage across  $L$  and the voltage across the resistance in series

with  $C$  are identical at all times. To prove this, substitute  $e = Ri = R(i_L + i_c)$  into Eq. (153).

$$R(i_L + i_c) = Ri_L + L \frac{di_L}{dt}$$

From this equation it is evident that

$$Ri_c \equiv L \frac{di_L}{dt}$$

In addition, the voltage across  $C$  and the voltage across the resistance in series with  $L$  are identical at all times. This can be demonstrated by substituting  $e = R(i_L + i_c)$  into Eq. (154).

$$R(i_L + i_c) = Ri_c + \frac{qc}{C}$$

From this equation it is concluded that

$$Ri_L \equiv \frac{qc}{C}$$

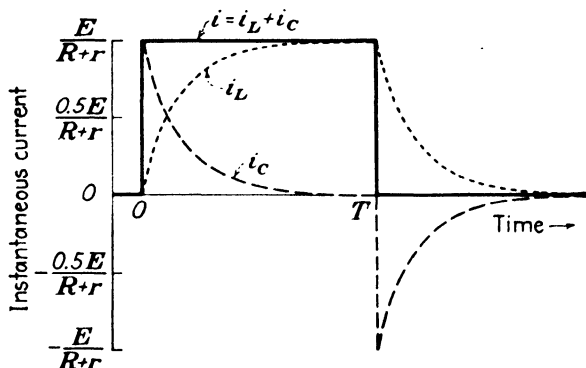


FIG. 158.—Behavior of instantaneous currents in the network of Fig. 156.

The equivalent network is valid for any generator voltage whatsoever. This can be verified by substituting a general voltage  $e_g$  for  $E$  in the foregoing proof without affecting the result. An important implication of this generality is in the case where  $e_g$  is an alternating voltage of variable frequency. For such a voltage, this network is "resonant" at all frequencies, including zero frequency. As far as the pulse-response characteristic is concerned, this network is capable of giving perfect

reproduction of pulse shape across the output, provided the values of the parameters satisfy the condition  $R = \sqrt{L/C}$ .

### CONCLUSION

The networks in this chapter have been selected from a vast variety of series-parallel networks containing  $R$ ,  $L$ , and  $C$ , and equivalent series networks have been found. This casts a favorable light on the scope of the equivalent-series-network method, but actually the majority of series-parallel *RLC* networks give rise to differential equations for charge that are of higher order than the second. Therefore, it should be emphasized that equivalent series networks exist in comparatively few instances.

## CHAPTER IX

### ELEMENTARY APPLICATIONS

In order to apply transient analysis of any kind to a real network, *i.e.*, one that exists physically, a very important step must be taken at the outset. Every element in the physical network must be considered individually and in relation to all other elements in order to determine the simplest configuration of parameters that will best represent the actual conditions. This important decision will determine how closely the analytical results describe the behavior of the physical network. The choice of parameters to represent a physical network is generally a compromise between exactness and simplicity.

In Chap. I it was mentioned that every physical element contains all three parameters,  $R$ ,  $L$ , and  $C$ , but that in many cases one or two of them may be justifiably neglected. Unfortunately, there is no set of rules that can be applied to discover which parameters can be neglected, and it is largely experience that governs the choice. Every network that is analyzed contains parameters that are based upon assumptions regarding the physical elements, and in practically all cases the assumptions are approximations. In general, an attempt is made to make assumptions that result in excellent agreement between the analytical results and the experimental observations in the physical network. In some cases, however, an analysis based upon assumptions that are known to be inaccurate is performed, and the results indicate only roughly the behavior of the physical network. This is done mainly in cases where a more accurate analysis is unduly complicated, and where the more accurate results do not represent a substantial gain in information over the less accurate analysis.

This chapter demonstrates the classical method as applied to several elementary physical networks. No attempt is made to include all possible applications. The intent is to demonstrate, by means of a few examples, the manner in which the

pulse-response characteristics of a real network can be found and the nature of the approximations that are required.

### DIODE DETECTOR

A diode-detector network, which can be used to produce a direct voltage proportional to the amplitude of a train of pulses, is shown in Fig. 159 and includes all stray network elements. A stray network element is one that is an inherent part of the network and that cannot be removed as an individual element.

**1. Simplification of Network.**—It would be a tremendous undertaking to analyze this network including all of the elements shown, but fortunately many of them play an insignificant role in determining the pulse-response characteristic. It is

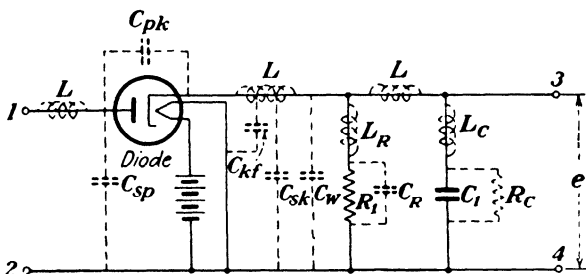


FIG. 159.—A diode-detector network in which all stray parameters are shown.

desirable to neglect as many elements as possible in order to simplify the analysis. For most practical purposes, many of the stray elements can be ignored completely. If the width of the pulses to be detected is not too short, then the following elements can usually be neglected:

1. The lead inductances  $L$ , provided the physical length of the leads is kept as short as possible.

2. The socket capacitance  $C_{sp}$ , because it is usually extremely small and cannot be lumped across  $C_1$  conveniently.

3. The inductance  $L_R$ , inherent in the resistor  $R_1$ , because this inductance is very small for most practical resistors.

4. The inductance  $L_C$ , inherent in the capacitor  $C_1$ . For the type of capacitor that would be used in this application, this neglect is usually justified. However, in some capacitors the inductance must be taken into account.

5. The plate-to-cathode capacitance  $C_{pk}$  of the diode. A diode with a low  $C_{pk}$  should be selected for this application.

The reason is not so that it may be omitted in the analysis, which is a helpful factor, but because a high  $C_{pk}$  might introduce an undesired reduction in output voltage.

In addition to these assumptions, the characteristics of the diode itself must be idealized so that is a "linear" element. Actually, the diode is not only a unidirectional device but also a nonlinear device. It will be assumed that the diode has infinite resistance (does not conduct electrons) when its plate voltage is equal to or less than its cathode voltage, and that it has a constant resistance  $R_d$  when its plate voltage exceeds its cathode voltage. This is usually a good approximation for generator-pulse voltages that are not extremely small in amplitude.

This completes the necessary assumptions as far as the network is concerned. However, no mention has been made, concerning the source of the pulses. It is assumed that the pulse source can be represented by a generator with internal resistance  $r$ . If the generator does not have an internal resistance only, the necessary parameters can be inserted in series with the generator. A rectangular generator pulse is assumed, although this network is relatively insensitive to pulse shape provided (1) the charging time constant is very much less than the discharging time constant, and (2) the charging time constant is small compared with the generator pulse width.

All of these assumptions have been incorporated in Fig. 160. When the switch is closed, it represents the condition under which the diode conducts, and when the switch is open, it represents the nonconducting condition.

**2. Pulse-response Characteristic.**—The analysis of the network in Fig. 160 can be made by the classical method. In fact, this network is essentially the same as that in Fig. 105, which was analyzed completely in Chap. VI. Suppose a single generator pulse is considered first. At the instant the pulse arrives, the diode begins to conduct; *i.e.*, the switch in Fig. 160 closes, and charge begins to accumulate on  $C$ . Charge continues to flow onto  $C$  during the pulse. When the generator pulse disappears, the diode becomes nonconducting; *i.e.*, the switch opens, and  $C$  discharges through  $R$ .

For purposes of analysis, Thévenin's theorem can be applied to the network in Fig. 160. The equivalent series network is

shown in Fig. 161. On the basis of this equivalent series network, the following behavior is evident:

1. The charge on  $C$  increases exponentially during the generator pulse.

$$\text{Charging time constant} = \frac{R(R_d + r)C}{R + R_d + r}$$

2. The steady-state value of output voltage during the generator pulse is  $ER/(R + R_d + r)$ .

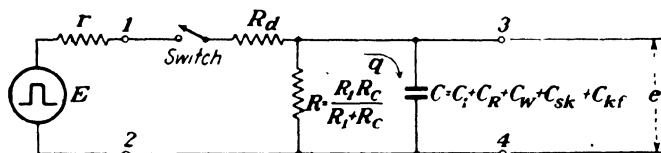


FIG. 160.—Equivalent series-parallel network for that in Fig. 159 after neglecting some of the stray parameters and assuming a "linear" diode.

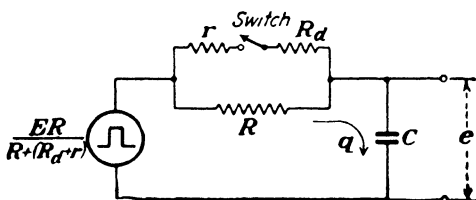


FIG. 161.—Equivalent series network for that in Fig. 160 as obtained by Thévenin's theorem. For either position of the switch, open or closed, the resistance can be regarded as a single equivalent resistance.

3. The charge on  $C$  decreases exponentially after the generator pulse disappears.

$$\text{Discharging time constant} = RC$$

4. The steady-state value of output voltage after the generator pulse disappears is zero.

The equations for the output voltage can be found in the same manner that Eqs. (117) and (121) were found in Chap. VI. They are

$$e_E = \frac{ER}{R + R_d + r} \left[ 1 - e^{-\frac{(R + R_d + r)t}{R(R_d + r)C}} \right] \quad (159)$$

$$e_0 = \frac{ER}{R + R_d + r} \left[ 1 - e^{-\frac{(R + R_d + r)T}{R(R_d + r)C}} \right] e^{-\frac{(t-T)}{RC}} \quad (160)$$

When the pulse width is large compared with the charging time constant, Eq. (160) simplifies to

$$e_0 = \frac{ER}{R + R_d + r} e^{-\frac{(t-T)}{RC}} \quad (160a)$$

Figure 162 shows the pulse-response characteristic of the network in Fig. 160 for a pulse width that is large compared with the charging time constant, and for a discharging time constant that is large compared with the charging time constant.

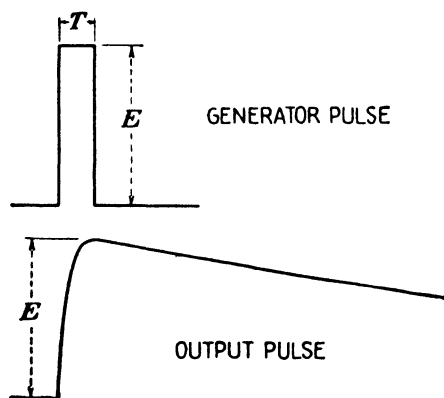


FIG. 162.—Output pulse of the diode-detector network when the pulse width is large compared with the charging time constant but small compared with the discharging time constant.

**3. Specific Example.**—The actual values of the elements in a diode-detector network depend upon the particular application. In most applications the relationship between the discharging time constant and the period of the train of pulses is of primary interest. In addition, an attempt is usually made to keep the charging time constant small compared with the pulse width. When the charging time constant is comparable with the discharging time constant, a train of generator pulses will appear in the output as discrete pulses if the pulse width is of the same order of magnitude or greater than the charging time constant. As the discharging time constant is made larger compared with the charging time constant, the output due to a train of generator pulses tends to become smoother, and in the practical limit there will be no trace of pulses in the output.

One set of values of network parameters that might be encountered is

$$\begin{array}{ll} E = 100 \text{ volts} & C_1 = 0.01 \text{ } \mu\text{f} \\ r = 250 \text{ ohms} & C_R = 1 \text{ } \mu\mu\text{f} \\ R_d = 450 \text{ ohms} & C_w = 3 \text{ } \mu\mu\text{f} \\ R_1 = 70,000 \text{ ohms} & C_{kf} = 4 \text{ } \mu\mu\text{f} \\ R_c = 10,000,000 \text{ ohms} & C_{sk} = 2 \text{ } \mu\mu\text{f} \end{array}$$

With these values,  $R$  and  $C$  in the network in Fig. 160 or Fig. 161 can be evaluated.

$$R = \frac{R_1 R_c}{R_1 + R_c} = \frac{R_1}{1 + (R_1/R_c)} \approx 70,000 \text{ ohms}$$

$$C = C_1 + C_R + C_w + C_{kf} + C_{sk} \approx 0.01 \text{ } \mu\text{f}$$

In this particular example  $R_1$ ,  $C_1$ ,  $r$ , and  $R_d$  largely determine the result. However, this is not always true. For instance, if  $C_1$  were of the order of  $10 \text{ } \mu\mu\text{f}$ , then the stray capacitances would have considerable influence.

The charging time constant is

$$\frac{R(R_d + r)C}{R + R_d + r} = \frac{RC}{1 + \frac{R}{R_d + r}} = \frac{70,000 \times 0.01 \times 10^{-6}}{1 + \frac{70,000}{700}} = 6.93 \text{ microseconds}$$

and the discharging time constant is

$$RC = 70,000 \times 0.01 \times 10^{-6} = 700 \text{ microseconds}$$

Thus the discharging time constant is approximately 100 times the charging time constant.

The steady-state value of output voltage during the generator pulse can be found from Eq. (159).

$$\frac{ER}{R + R_d + r} = \frac{100 \times 70,000}{70,000 + 700} = 99 \text{ volts}$$

If the generator-pulse width is 30 microseconds, for example, then this output voltage is very nearly attained because the pulse width is more than four times the charging time constant. The pulse response in Fig. 162 was derived for a pulse width of 30 microseconds and the value listed above.

**4. Repetitive Generator Pulses.**—The analysis for a single generator pulse can be extended to the case of a train of pulses.

However, unless the discharging time constant is small compared with the period, or time separation, of the generator pulses, a precise analytical solution is difficult. Nevertheless, a graphical solution can be made. Suppose, for instance, that pulses of 30-microsecond duration start every 200 microseconds as shown in Fig. 163, and that they are applied to a diode-detector network that has the previously listed values of parameters. The output voltage is also given in Fig. 163. Since the period of the applied pulses, 200 microseconds, is small compared with the discharging time constant, 700 microseconds, the transient due to a given pulse is very large when the next pulse arrives.

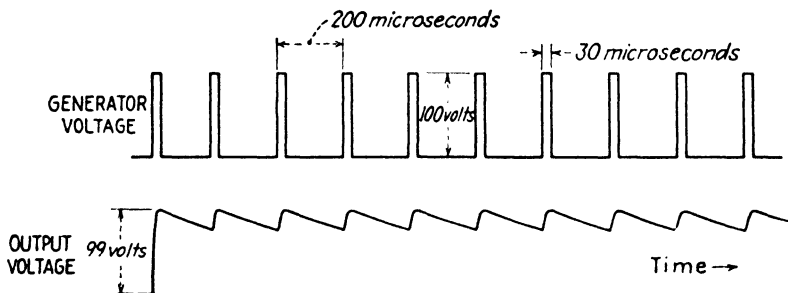


FIG. 163.—Output voltage of the diode-detector network for a train of rectangular-generator pulses. The pulse width is large compared with the charging time constant, and the pulse period is small compared with the discharging time constant. On the first rise, the output voltage reaches 99 volts. Subsequent maximum output voltages are approximately 100 volts.

The output voltage is approximately saw-tooth in shape. Its average value can be computed in an approximate manner by averaging the maximum and minimum values. The maximum value will be 100 volts, and the minimum value can be computed from Eq. (160a). A similar shape of output voltage would result for any 30-microsecond pulse irrespective of its shape.

There are other interesting features of this diode-detector network that can be analyzed in an approximate manner. Some of them are expressed in the form of problems at the end of this chapter. However, it is not the primary intent to analyze a specific network in great detail but rather to examine the fundamental considerations. Of principal importance is the marked difference between the physical network and the simplified network that is used in the analysis. For the particular

values of parameters that were used in the example, the results are quite accurate, and it would have been wasted effort to include all of the stray network parameters. However, in some cases the approximations introduced to obtain the network in Fig. 160 would lead to results that depart considerably from the actual network behavior.

### TUNED AMPLIFIER

In many instances the classical method is useful merely to explain the operation of certain networks. An amplifier that contains a tuned plate circuit provides a good example of this. While a detailed analysis could be performed, this section is intended to give a qualitative explanation only, and several approximations will be made that would not be justified in a mathematical analysis.

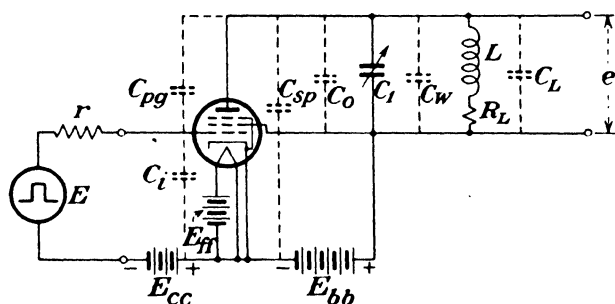


FIG. 164.—A pentode amplifier that contains a tuned plate circuit and a rectangular-pulse grid voltage. Most of the significant stray elements are included.

Figure 164 shows an amplifier that has sufficient fixed voltage  $E_{cc}$  in its grid circuit to cut off the flow of plate current. The plate circuit consists of a coil of inductance  $L$ , which contains, inherently, a small amount of resistance represented by  $R_L$ .  $C_0$  is the output capacitance of the tube,  $C_L$  is the distributed capacitance of the coil,  $C_w$  is the wiring capacitance,  $C_{sp}$  is the plate socket capacitance, and  $C_1$  is a variable "tuning" capacitor. It is assumed that the lead inductance is negligible. In most applications the resistance of the coil is very small, and it has been neglected in Fig. 165 for the purpose of this qualitative analysis. The plate-to-control grid capacitance  $C_{pg}$  has also

been neglected in Fig. 165. The generator pulse is inverted in the plate circuit because the plate voltage decreases when the control grid voltage is driven above cutoff by the rectangular pulse. The resistance  $r_p$ , which represents the dynamic plate resistance of the pentode, is really not a fixed resistance since the grid voltage varies over such a wide range. However,  $r_p$  is very large for a pentode, so, as a rough approximation, it can be assumed to be a large fixed value.

**5. Pulse-response Characteristic.**—The network in Fig. 165 can be regarded as a series  $RLC$  network insofar as the pulse-response characteristic is concerned, as was shown in Chap. VIII. (See Figs. 154a and 155.) The criterion for the form of the pulse-response characteristic can be found on the basis of the equivalent series network. The relative value of  $1/r_p$  compared with  $2\sqrt{C/L}$  turns out to be the critical factor. Since  $1/r_p$  is always small compared with  $2\sqrt{C/L}$  in this particular application, then the condition for an oscillatory output is fulfilled.

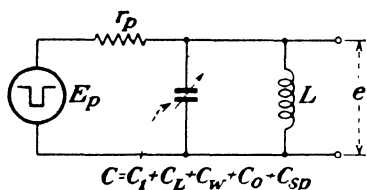


FIG. 165.—Approximate equivalent network for that in Fig. 164 after neglecting many factors. D-c elements are omitted.

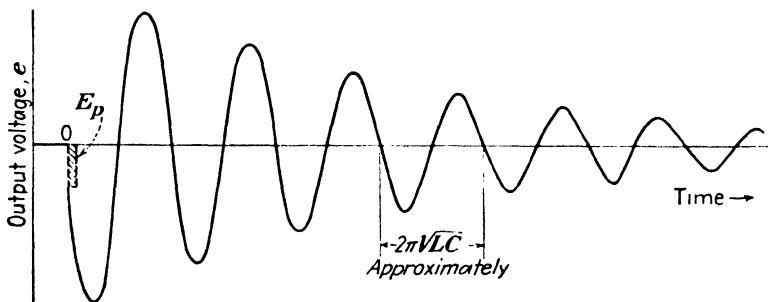


FIG. 166.—Output voltage of the tuned amplifier for a single rectangular pulse.

Figure 166 shows the output voltage for the case of a single rectangular pulse. The pulse width is small compared with the period of the oscillatory output. The period is approximately  $2\pi\sqrt{LC}$  [see Eq. (87), Chap. V] since  $R_L$  has been neglected and  $r_p$  is very large. The damping is gradual for the same reasons.

The single pulse causes a burst of energy to flow into the plate circuit, and this energy is exchanged in an oscillatory manner between  $L$  and  $C$ . However, the amount of energy in the plate circuit gradually decreases because of the fact that  $r_p$  and  $R_L$  (which has been neglected) dissipate energy. The actual source of energy is the plate supply battery  $E_{bb}$ , and the control grid functions as a valve that permits energy to flow from  $E_{bb}$  into the plate circuit during the pulse.

**6. Repetitive Generator Pulses.**—When a train of pulses is applied to the grid of the amplifier, the transient due to one pulse does not have time to diminish to a negligible value before

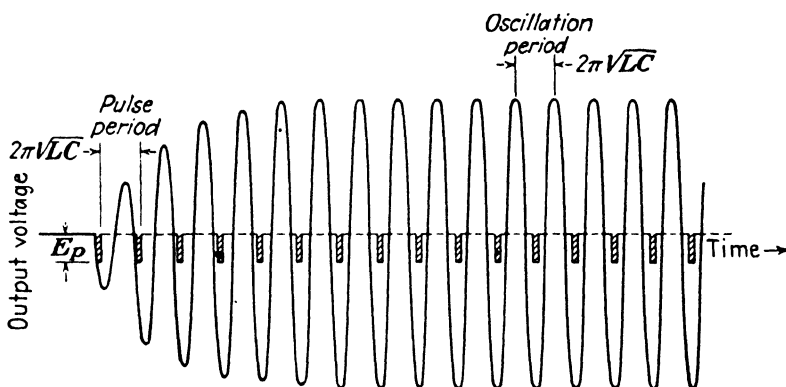


FIG. 167.—Output voltage of the tuned amplifier when a train of generator pulses is applied. The amplifier plate circuit is tuned to the frequency of the applied pulses.

the subsequent pulse arrives, unless the pulse period is extremely large compared with the natural or freely oscillating period of the plate circuit. This property is taken advantage of in two important applications. In one, the tuning capacitor  $C_1$  is adjusted so that the free period of oscillation of the plate circuit is exactly equal to the period of the pulses applied to the control grid. When the amplifier is used in this manner, it is called a *Class C* amplifier. In practice, the pulses on the control grid are derived from the peaks of a sinusoidal signal that has the same period as the natural period of the plate circuit. These peaks are often assumed to be approximately rectangular for ease of analysis. A second common application of this network is as a frequency multiplier. When used as a frequency doubler, the tuning capacitor is adjusted so that the free period of oscilla-

tion in the plate circuit is exactly equal to one-half the period of the applied pulses; in other words, the plate-circuit natural frequency is tuned to twice the frequency of the applied pulses.

*Class C Amplifier.*—Figure 167 shows the output voltage for the case when the plate circuit is tuned to the frequency of the applied pulses. From this figure it can be seen that equilibrium is reached, after a short transitional period, and the output voltage consists of a sustained sinusoidal oscillation that has a period equal to the natural or free period of oscillation

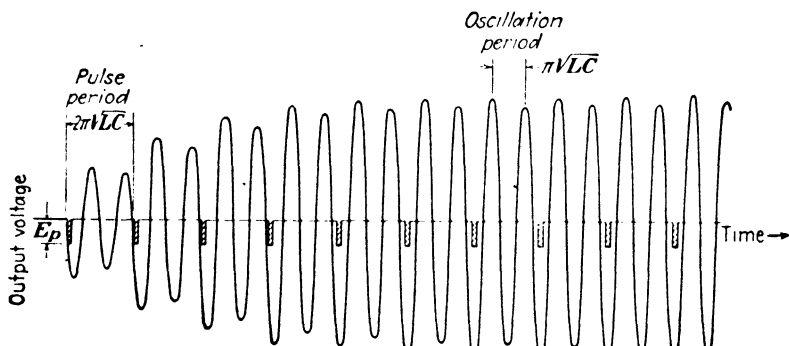


FIG. 168.—Output voltage of the tuned amplifier when the plate circuit is tuned to twice the frequency of the applied pulses.

of the plate circuit. The pulse occurs at just the right instant in the sinusoidal cycle to reenforce the oscillation. Since the natural damping in the plate circuit is not great, the output voltage is very nearly a perfect sine wave.

*Frequency Doubler.*—If the tuning capacitor is adjusted so that the plate circuit is tuned to twice the frequency of the applied pulses, the output voltage will again consist of a sustained oscillation. This condition is shown in Fig. 168. The output voltage has a period that is determined mainly by the parameters  $L$  and  $C$ ; this period is one-half the period of the applied pulses. Again the pulse occurs at just the right instant in the sinusoidal cycle to reenforce the oscillation. Notice the slight effect due to damping because in this case two full cycles of free oscillation must take place before the output is reenforced by a succeeding pulse.

*Energy Considerations.*—The oscillation in the plate circuit is due, fundamentally, to an exchange of energy between  $L$  and

$C$ ; the source of this energy is the plate supply battery. For a single pulse, the oscillation endures for a comparatively short time and gradually dies out because the energy supplied during one pulse is eventually dissipated in  $r_p$  and  $R_L$ . However, when a train of pulses exists, energy is supplied periodically to the oscillating circuit. There is an initial build-up of the amount of energy involved, and subsequently an equilibrium is reached when the amount of energy supplied to the oscillating circuit is exactly equal to the amount of energy dissipated in it between bursts of supplied energy.

### SIMPLE FILTERS

In the examples given so far, the pulse-response characteristics have been used to good advantage. In the case of the diode detector an output voltage was produced that was proportional to the pulse voltage and hence contained useful information. In the case of the tuned amplifier, advantage was taken of an oscillatory condition to produce a sustained sinusoidal oscillation from a train of pulses. However, in some cases the presence of an output voltage that is due to pulses is not desired. Many networks have been designed specifically to eliminate from their output any voltage due to pulses at the network input. Such networks are called *filters*.

When the voltage due to a pulse is not desired in the output of a network, this does not mean that a knowledge of its pulse-response characteristic is unimportant. On the contrary, advantage must be taken of the pulse-response characteristics so that the output pulse is minimized, and a knowledge of the transient response of networks is required to get rid of the output voltage as well as to utilize an output voltage that results from a pulse input.

The network in Fig. 169 shows a source of direct voltage, represented by a battery of internal resistance  $r$ , that is delivering current to a load consisting of a pure resistance  $R$ . The source of the direct voltage might actually be a dynamo (d-c generator),

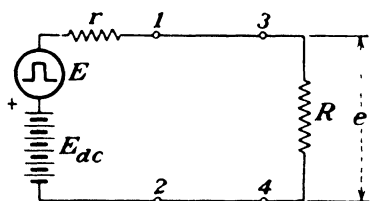


FIG. 169.—D-c generator that is delivering power to a load resistance  $R$ .  $E$  is a pulse originating within the generator.

vibrator power supply, or vacuum-tube power supply, to name a few. In addition, a rectangular pulse voltage exists that originates somewhere within the source of the direct voltage and cannot be eliminated from the generator. The appearance of a pulse across the load  $R$  is assumed to be particularly objectionable. A means of minimizing the voltage across  $R$  that is due to the rectangular pulse must be found. At the same time the d-c conditions must be preserved as nearly as practicable.

If a network that has a suitable pulse-response characteristic is placed between the generator and the load, the undesired pulse voltage can be almost completely eliminated from the load with little sacrifice in direct voltage. The simple networks shown in Fig. 170 can be used in this application. Since the

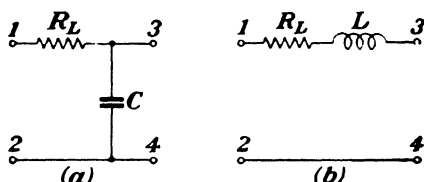


FIG. 170.—Two filters that can be used to reduce the pulse voltage appearing across the load in the network of Fig. 169.

filtering properties of these two networks are going to be compared, the series resistance  $R_L$  is assumed to be the same in both networks. This is done in order to make them equivalent as far as the direct current is concerned, since in both cases the load current will flow between the terminals 1-3. In other words, the same direct voltage drop will occur across each filter when it is inserted between the load and the generator.

It is important to realize that certain assumptions have already been made. Some of them are

1. The lead inductance is negligible in all cases.
2. The inductance inherent in  $C$  is negligible.
3. The distributed capacitance of  $L$  and of the resistor in the network in Fig. 170a is negligible.

4. The capacitance across terminals 1-2 and 3-4 is negligible. As a matter of fact, all the elements have been assumed to be ideal with the exception of the inductor  $L$  whose inherent resistance  $R_L$  has been taken into account. These assumptions are usually justified for this application.

**7. Equations for Output Pulse.**—Since the filters are equivalent as far as direct current is concerned, it is possible to neglect the presence of the battery for the purpose of a pulse analysis. To compare these two filters, they can be inserted between the source of the rectangular pulse and the load as shown in Fig. 171.

**RC Filter.**—The pulse-response characteristic of the network in Fig. 171a can be determined indirectly by finding the equivalent series network. The differential equation for the load

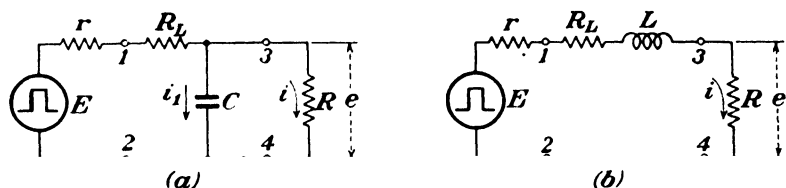


FIG. 171.—Connection of filters between the generator and the load. The d-c generator has been omitted for the purpose of the pulse analysis.

current  $i$  in terms of  $t$  only can be found by applying Kirchhoff's laws to the network. Two independent voltage equations are

$$E = (R_L + r)(i_1 + i) + Ri \quad (161)$$

$$\frac{q_1}{C} = Ri \quad (162)$$

To eliminate  $i_1$  from Eq. (161), solve Eq. (162) for  $q_1$  and differentiate.

$$q_1 = RCi$$

$$\frac{dq_1}{dt} = i_1 = RC \frac{di}{dt}$$

Substituting this value of  $i_1$  into Eq. (161) results in the desired differential equation for  $i$  in terms of time only.

$$E = R(R_L + r)C \frac{di}{dt} + (R + R_L + r)i$$

The coefficients of this equation determine the parameters of the equivalent series network given in Fig. 172. This equivalent network enables a remarkable comparison to be made between the two filters: the equivalent series network and the network in Fig. 171b will be identical if  $L_s = R(R_L + r)C$  is equal to  $L$ .

The equations for the output voltage can be found on the basis of the equivalent series network, which is the same as the network in Fig. 46, Chap. IV. The output-voltage equations are

$$e_E = \frac{ER}{R + R_L + r} [1 - e^{-\frac{(R + R_L + r)t}{R(R_L + r)C}}] \quad (163)$$

$$e_0 = \frac{ER}{R + R_L + r} [e^{\frac{(R + R_L + r)T}{R(R_L + r)C}} - 1] e^{-\frac{(R + R_L + r)t}{R(R_L + r)C}} \quad (164)$$

**L Filter.**—The equations for output voltage for the network in Fig. 171b are of the same form as Eqs. (54) and (57) which

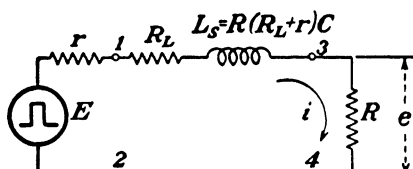


FIG. 172.—Equivalent series network for that in Fig. 171a. Note that the two filters in Fig. 171 are basically the same.

were derived in Chap. IV for the same type of network. They are

$$e_E = \frac{ER}{R + R_L + r} [1 - e^{-\frac{(R + R_L + r)t}{L}}] \quad (165)$$

$$e_0 = \frac{ER}{R + R_L + r} [e^{\frac{(R + R_L + r)T}{L}} - 1] e^{-\frac{(R + R_L + r)t}{L}} \quad (166)$$

These equations are essentially the same as Eqs. (163) and (164) since the equivalent series network is basically the same as the L-filter network.

**8. Filtering Properties.**—The equations for output voltage show that each network will have an identical pulse-response characteristic provided

$$L_s = R(R_L + r)C = L$$

Therefore, either filter can be equally effective in minimizing the voltage pulse that appears across the load resistance  $R$ .

A specific example will serve to demonstrate the effectiveness of the filter. Suppose the direct voltage of the generator is 200 volts, the internal resistance is 20 ohms, the load resistance is 2,000 ohms, the pulse voltage that appears across the load

resistance is 99 volts when no filter is employed, and the pulse width is 15 microseconds. Figure 173 illustrates the load voltage as a function of time for these particular values.

The direct load voltage is 198 volts because there is a 2-volt drop across  $R$  due to the direct current.

$$\text{Direct load voltage} = \frac{E_{dc}R}{R + r} = \frac{200 \times 2,000}{2,020} = 198 \text{ volts}$$

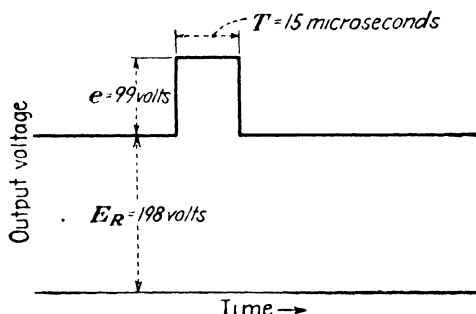


FIG. 173.—Load voltage of the network in Fig. 169 with no filter.

Since the pulse voltage across the load resistance is 99 volts, the generator voltage is 100 volts.

$$\text{Pulse load voltage} = \frac{ER}{R + r} = \frac{2,000E}{2,020} = 99$$

$$E = \frac{2,020 \times 99}{2,000} = 100 \text{ volts}$$

A suitable criterion for effectiveness of filtering can be defined in percentage by the ratio

$$F = \frac{\text{peak instantaneous load voltage due to pulse alone}}{\text{direct load voltage}} \times 100$$

This ratio will be zero for perfect filtering. In the case of no filter at all, the ratio is

$$F = \frac{99}{198} \times 100 = 50 \text{ per cent}$$

Now suppose the filter in Fig. 170a with  $R_L = 180$  ohms and  $C = 0.25 \mu f$  is introduced between the generator and the

load as shown in Fig. 174. The same filtering will take place if the filter in Fig. 170b is used with a value of

$$L = R(R_L + r)C = 2,000(200)0.25 \times 10^{-6} = 0.1 \text{ henry}$$

and  $R_L = 180 \text{ ohms}$

To evaluate the ratio  $F$ , first the direct load voltage can be computed. It will be less than 198 volts now, due to the addition of the 180-ohm series resistor.

$$\text{Direct load voltage} = \frac{E_{dc}R}{R + R_L + r} = \frac{200 \times 2,000}{2,200} = 182 \text{ volts}$$

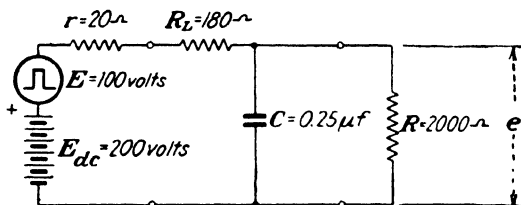


FIG. 174.—Numerical example of an  $RC$  filter network.  $F = 14.1$  per cent.

The peak instantaneous load voltage due to a generator pulse of 100 volts can be found from Eq. (163) or (165). The steady-state value of output pulse during the generator pulse is

$$\frac{ER}{R + R_L + r} = \frac{100 \times 2,000}{2,200} = 91 \text{ volts}$$

and the time constant is

$$\frac{L}{R + R_L + r} = \frac{R(R_L + r)C}{R + R_L + r} = \frac{2,000 \times 200 \times 0.25 \times 10^{-6}}{2,200} = 45.5 \text{ microseconds}$$

The peak output voltage occurs at  $t = T$ .

$$(e_E)_T = 91(1 - e^{-\frac{15}{45.5}}) = 91(1 - e^{-0.33}) = 25.6 \text{ volts}$$

Therefore, the ratio  $F$  is

$$F = \frac{25.6}{182} \times 100 = 14.1 \text{ per cent}$$

If a greater reduction in pulse voltage is desired, it is necessary to increase the network time constant. As the ratio of

the pulse width to the time constant approaches zero,  $F$  will approach zero. Suppose  $C$  is increased by a factor of four to a value of  $1.0 \mu\text{f}$ . Then the ratio of pulse width to time constant is decreased by a factor of four: from 0.33 to 0.082. The time constant with  $C = 1.0 \mu\text{f}$  is

$$45.5 \times \frac{1 \times 10^{-6}}{0.25 \times 10^{-4}} = 182 \text{ microseconds}$$

The peak pulse voltage is then

$$(e_E)_T = 91(1 - e^{-0.082}) = 91(1 - 0.92) = 7.28 \text{ volts}$$

and the filtering ratio becomes

$$F = \frac{7.28}{182} \times 100 = 4 \text{ per cent}$$

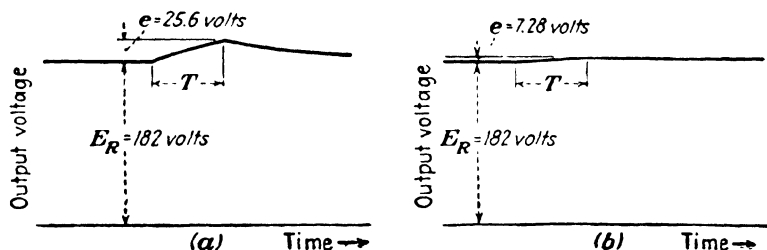


FIG. 175. Load voltage of the network in Fig. 174 for two filters with different time constants. In (a),  $C = 0.25 \mu\text{f}$  as shown in Fig. 174. In (b),  $C = 1.0 \mu\text{f}$ .  $T = 15$  microseconds.

The load voltage in each of these cases is shown in Fig. 175. The pulse output in Fig. 175b is approximately the integral of the generator pulse since the network time constant is very large compared with the pulse width.

Before concluding this section it is well to point out that the two filters presented will give comparable filtering for any pulse shape, provided the network time constant is very large compared with the pulse width. There are other filter networks that give much smaller values of the filtering ratio, and the two that have been discussed were presented mainly as simple examples that demonstrate the fundamental principles.

### PULSE AMPLIFIER

It is sometimes desired to increase the voltage of small rectangular pulses. This is commonly done by means of a vacuum-

tube amplifier. A resistance-capacitance coupled amplifier will serve this purpose if the reproduction of pulse shape is not an important factor. However, if it is necessary to amplify the pulse with minimum change in shape, a video amplifier is superior.<sup>1</sup>

The resistance-capacitance coupled amplifier shown in Fig. 176 will be analyzed. A small positive rectangular voltage  $E$  of duration  $T$  is applied to the control grid of the amplifier tube. This pulse will appear in the plate circuit, and the voltage across  $R$  is the output voltage. The output can be applied to the grid of a succeeding amplifier as shown. The object of this analysis is twofold: (1) to determine the amplification that takes place;

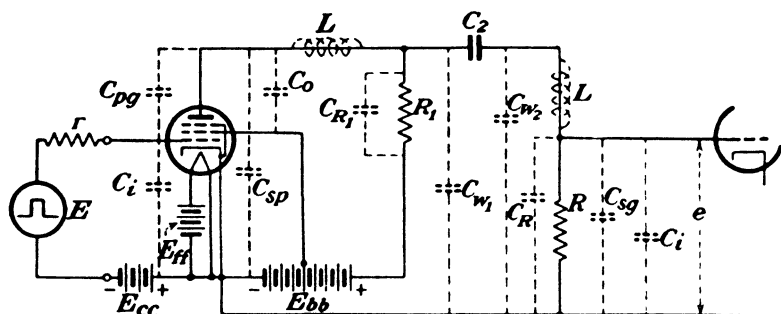


FIG. 176.—Resistance-capacitance coupled amplifier in which most of the stray parameters are shown. The shunt resistance inherent in  $C_2$  is neglected.

*i.e.*, the ratio of the output pulse amplitude to the input pulse amplitude, and (2) to determine the shape of the output pulse.

**9. Simplification of Network.**—Before the analysis can be performed, the inescapable job of simplifying the network by neglecting factors that are insignificant in determining the pulse-response characteristic must be performed. The following assumptions are usually justified:

1. The lead inductances  $L$  are negligible provided the leads are kept as short as possible.
2. The plate-to-control grid capacitance  $C_{pg}$  of the pentode amplifier is negligible.
3. The generator internal resistance  $r$  is so small that the input capacitance  $C_i$  connected across the generator is negligible.

<sup>1</sup> D. G. Fink, "Principles of Television Engineering," 1st ed., Chap. VI, McGraw-Hill Book Company, Inc., New York, 1940.

4. The pulse voltage applied to the control grid of the amplifier is so small that the amplifier operates in a linear manner.

Other assumptions could be made, but no further simplification of the network would result, so no other parameters will be omitted; *i.e.*,  $C_0$  and  $C_i$  cannot be neglected, and  $C_{sp}$ ,  $C_{R1}$ , and  $C_{w1}$  can be lumped across  $C_0$  while  $C_{so}$ ,  $C_R$ , and  $C_{w2}$  can be lumped across  $C_i$ .

These assumptions are represented by the equivalent plate circuit in Fig. 177a, where  $r_p$  is the dynamic plate resistance of the tube and  $\mu$  is its control grid-to-plate amplification factor.<sup>1</sup>

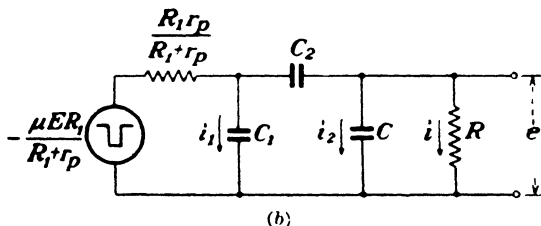
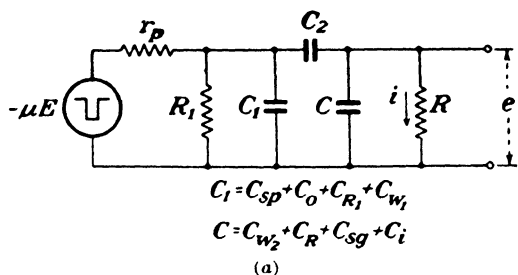


FIG. 177.—Equivalent series-parallel networks for the amplifier circuit in Fig. 176 after neglecting some of the stray parameters. D-c elements are omitted.

Since the pulse-response characteristic is of interest, all d-c elements have been omitted. The negative generator voltage merely signifies that the output pulse will be negative for an applied pulse that is positive.

**10. Pulse-response Characteristic.**—Application of Thévenin's theorem to the network in Fig. 177a reduces it to that shown in Fig. 177b. To determine the pulse-response characteristic of this network it is convenient to find the equivalent series

<sup>1</sup> For a treatment of the equivalent plate circuit of amplifiers refer to J. Millman and S. Seely, "Electronics," Chap. XVII, McGraw-Hill Book Company, Inc., New York, 1941.

network. This can be done by obtaining the differential equation for  $i$  in terms of time only. Kirchhoff's laws enable the following voltage equations to be written:

$$\frac{-\mu ER_1}{R_1 + r_p} = \frac{R_1 r_p}{R_1 + r_p} (i_1 + i_2 + i) + \frac{q + q_2}{C_2} + Ri \quad (167)$$

$$\frac{q_2}{C} = Ri \quad (168)$$

$$\frac{q_1}{C_1} = \frac{q + q_2}{C_2} + Ri \quad (169)$$

To eliminate all variables except  $i$  and  $t$ , proceed as follows: From Eq. (168),  $q_2 = RCi$  and

$$\frac{dq_2}{dt} = i_2 = RC \frac{di}{dt}$$

From Eq. (169)

$$q_1 = \frac{C_1}{C_2} (q + q_2) + RC_1 i$$

Substitute  $q_2 = RCi$  for  $q_2$ .

$$q_1 = \frac{C_1}{C_2} (q + RCi) + RC_1 i$$

Differentiate.

$$\frac{dq_1}{dt} = i_1 = \frac{C_1}{C_2} \left( i + RC \frac{di}{dt} \right) + RC_1 \frac{di}{dt}$$

Substitute these relations into Eq. (167).

$$\begin{aligned} \frac{-\mu ER_1}{R_1 + r_p} = \frac{R_1 r_p}{R_1 + r_p} \left[ \frac{C_1}{C_2} \left( i + RC \frac{di}{dt} \right) + RC_1 \frac{di}{dt} + RC \frac{di}{dt} + i \right] \\ + \frac{q + RCi}{C_2} + Ri \end{aligned}$$

Collect like terms.

$$\begin{aligned} \frac{-\mu ER_1}{R_1 + r_p} = \left[ \frac{R_1 r_p}{R_1 + r_p} \left( \frac{CC_1}{C_2} + C + C_1 \right) \right] \frac{di}{dt} \\ + \left[ \frac{R_1 r_p}{R_1 + r_p} \left( \frac{C_1}{C_2} + 1 \right) + \frac{RC}{C_2} + R \right] i + \frac{q}{C_2} \end{aligned}$$

The coefficients of this differential equation determine the parameters of an equivalent series network. The equivalent series parameters are

$$\begin{aligned} E_s &= \frac{-\mu E R_1}{R_1 + r_p} \\ L_s &= \frac{R_1 R r_p}{R_1 + r_p} \left( \frac{C C_1}{C_2} + C + C_1 \right) \\ R_s &= R + \frac{R C}{C_2} + \frac{R_1 r_p}{R_1 + r_p} \left( \frac{C_1}{C_2} + 1 \right) \\ C_s &= C_2 \end{aligned}$$

Figure 178 shows the equivalent series network that has the same pulse-response characteristic as those in Fig. 177.

Since the actual network is assumed to contain no inductance, then the equivalent network in Fig. 178 must be overdamped,

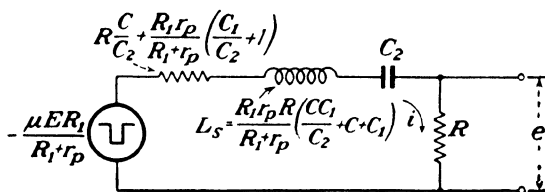


FIG. 178.—Equivalent series network for those in Fig. 177.

i.e.,  $M_s^2 > N_s$ . The equations for output voltage both during and after the generator pulse can be found as in Chap. V, but because such a network has already been analyzed, advantage can be taken of the equations already derived. Equations (94) and (101) pertain to such a network for the case where  $M^2 > N$ . If the values of the network parameters  $R_s$ ,  $L_s$ , and  $C_s$  are substituted for  $(R + r)$ ,  $L$ , and  $C$  in the network in Fig. 82, Chap. V, then the following equations can be written directly:

$$e_E = \frac{2E_s R M_s e^{-M_s t}}{R_s \sqrt{M_s^2 - N_s}} \sinh \sqrt{M_s^2 - N_s} t \quad (94')$$

$$e_0 = \frac{-2E_s R M_s e^{-M_s t}}{R_s \sqrt{M_s^2 - N_s}} \cdot \left[ \epsilon^{M_s T} \sinh \sqrt{M_s^2 - N_s} (t - T) - \sinh \sqrt{M_s^2 - N_s} t \right] \quad (101')$$

**11. Specific Example.**—For a given amplifier tube, there are essentially three parameters over which control can be exercised;

namely,  $R_1$ ,  $C_2$ , and  $R$ , because  $\mu$ ,  $r_p$ ,  $C_0$ , and  $C_i$  are determined by the tube, and  $C_{sp}$ ,  $C_{R1}$ ,  $C_{w1}$ ,  $C_{sg}$ ,  $C_R$ , and  $C_{w2}$  can be held to a small minimum value in most cases. Suppose a given amplifier circuit has the following values:

$$\begin{aligned} r_p &= 10^6 \text{ ohms} & C_{w1} &= C_{w2} = 2 \mu\text{mf} \\ \mu &= 1,500 & C_2 &= 1,000 \mu\text{mf} \\ C_0 = C_i &= 6 \mu\text{mf} & R_1 &= 10^4 \text{ ohms} \\ C_{sp} = C_{sg} = C_{R1} = C_R &= 1 \mu\text{mf} & R &= 10^5 \text{ ohms} \end{aligned}$$

Then  $C_1$  and  $C$  in the networks in Fig. 177 are each equal to  $10 \mu\text{mf}$  or  $10^{-11}$  farad.

The values of the network parameters and other network constants can now be computed. It is usually wasted effort to compute values to accuracy that is better than 1 per cent since the network elements are usually known to an accuracy of only 5 to 20 per cent.

$$\begin{aligned} E_s &= \frac{-\mu E R_1}{R_1 + r_p} = \frac{-\mu E (R_1/r_p)}{1 + (R_1/r_p)} \\ &= \frac{-1.5 \times 10^3 \times E \times (10^4/10^6)}{1 + (10^4/10^6)} = \frac{-15E}{1.01} \approx -15E \end{aligned}$$

$$\begin{aligned} L_s &= \frac{R_1 R r_p}{R_1 + r_p} \left( \frac{C C_1}{C_2} + C + C_1 \right) = \frac{R_1 R C}{1 + (R_1/r_p)} \left( 2 + \frac{C_1}{C_2} \right) \\ &= \frac{10^4 \times 10^5 \times 10^{-11}}{1 + (10^4/10^6)} \left( 2 + \frac{10^{-11}}{10^{-9}} \right) \approx 0.02 \text{ henry} \end{aligned}$$

$$C_s = C_2 = 1,000 \mu\text{mf} = 10^{-9} \text{ farad}$$

$$\begin{aligned} R_s &= R + R \frac{C}{C_2} + \frac{R_1}{(R_1/r_p) + 1} \left( \frac{C_1}{C_2} + 1 \right) \\ &= 10^5 + \frac{10^5 \times 10^{-11}}{10^{-9}} + \frac{10^4}{1.01} \left( \frac{10^{-11}}{10^{-9}} + 1 \right) \approx 11 \times 10^4 \text{ ohms} \end{aligned}$$

$$M_s = \frac{R_s}{2L_s} = \frac{11 \times 10^4}{2 \times 0.02} = 2.75 \times 10^6$$

$$M_s^2 = (2.75)^2 \times 10^{12} = 7.56 \times 10^{12}$$

$$N_s = \frac{1}{L_s C_s} = \frac{1}{0.02 \times 10^{-9}} = 5 \times 10^{10}$$

$$\frac{M_s^2}{N_s} = \frac{7.56 \times 10^{12}}{5 \times 10^{10}} = 151$$

$$\alpha_s = \tanh^{-1} \sqrt{1 - \frac{N_s}{M_s^2}} = \tanh^{-1} \sqrt{1 - \frac{1}{151}}$$

$$= \tanh^{-1} 0.997 \approx 3.2$$

Since  $M_s^2$  is 151 times  $N_s$ , the network is considerably overdamped. Recall that the output voltage of the network in Fig. 178 will go through a maximum during the generator pulse

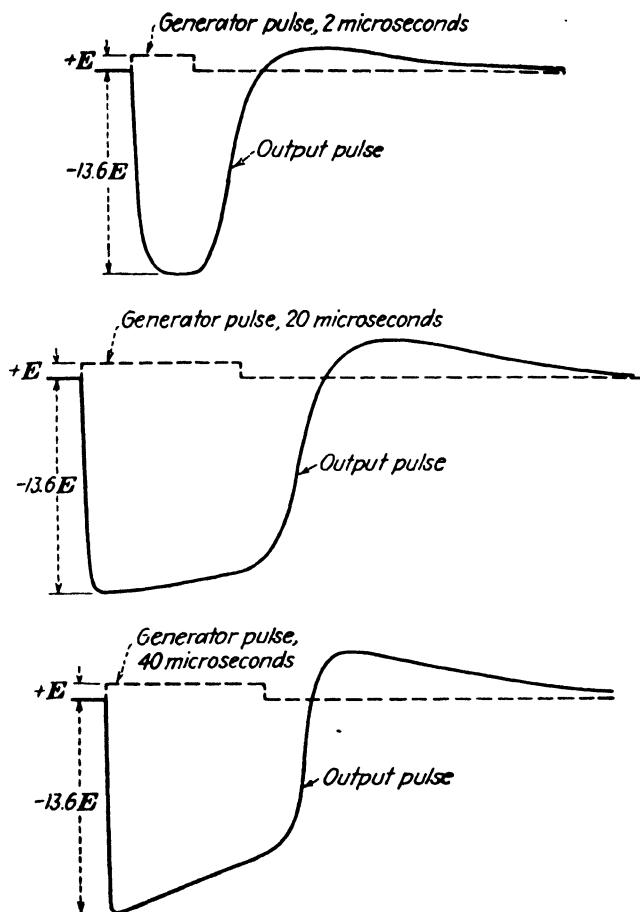


FIG. 179.—Output pulses of the amplifier in Fig. 176 for three different generator-pulse widths of equal amplitude  $E$ .

if the pulse width permits. (See Fig. 83.) The time at which this maximum occurs is given by Eq. (99) in Chap. V.

$$t' = \frac{\alpha_s}{\sqrt{M_s^2 - N_s}} = \frac{\alpha_s}{M_s \sqrt{1 - (N_s/M_s^2)}} \approx \frac{\alpha_s}{M_s} = \frac{3.2}{2.75 \times 10^6} = 1.16 \text{ microseconds}$$

This means that the pulse to be amplified must be at least 1.16 microseconds in width if the output voltage is to reach its maximum value. The maximum value of output voltage is given by Eq. (100), Chap. V.

$$\begin{aligned} e_{E\max} &= \frac{2E_s R M_s}{R_s \sqrt{N_s}} e^{\frac{M_s a s}{\sqrt{M_s^2 - N_s}}} \approx \frac{2E_s R M_s}{R_s \sqrt{N_s}} e^{-as} \\ &= \frac{-2 \times 15E \times 10^5 \times 2.75 \times 10^6}{11 \times 10^4 \times 2.24 \times 10^5} e^{-3.2} = -13.7E \end{aligned}$$

Therefore, the magnitude of the voltage gain of the amplifier is

$$\frac{\text{Peak output voltage}}{\text{Peak input voltage}} = \frac{13.7E}{E} = 13.7$$

The output pulse is shown in Fig. 179 for several different pulse widths. The time that is necessary for the output to go from zero to one half its maximum value is the time delay, and is approximately 0.12 microsecond for this amplifier.

### CONCLUSION

The illustrations in this chapter represent only a few instances in which the classical method can be utilized for a pulse analysis of practical networks. There are many other interesting examples. However, the significant steps in the analytical procedure have been covered, and they should serve as a guide for the pulse analysis of other networks. These steps are

1. Examine the physical network and draw a complete circuit that includes all stray elements.
2. On the basis of experience, common sense, etc., omit all elements that will have negligible influence upon the pulse-response characteristic.
3. Analyze the simplified network directly, or by converting it to an equivalent series network.

In retrospect, it can be seen that Chaps. III, IV, and V contain the basic and underlying analyses of all of the networks discussed in this book. Subsequent chapters have served to show that the equations in Chaps. III, IV, and V are not necessarily confined to the particular networks from which they were derived. A similar pattern is found in the study of most engineering subjects; a relatively small amount of

information opens the way to a field of numerous and diverse applications.

### Problems

**Prob. 1.** Using the values given on page 235 for the diode detector in Fig. 159, compute the approximate average value of the output voltage for a pulse width of 50 microseconds and a pulse period of 150 microseconds.

**Prob. 2.** In Prob. 1, suppose generator pulses are alternately 100 volts and 80 volts in amplitude. It is desired to have the output voltage follow this variation in pulse amplitude; *i.e.*, the voltage across  $C$  must decay to 80 volts by the time an 80-volt pulse arrives at the input.

- If  $R_1$  only is changed, what new resistance is required?
- If  $C_1$  only is changed, what new value of capacitance is required?
- Draw a sketch of the output voltage.

**Prob. 3.** It is desired to reduce the output voltage ripple shown in Fig. 163. Using the values given on page 235, what additional capacitance must be connected in parallel with  $C_1$  in order to achieve a difference of 10 volts between the maximum and minimum values of output voltage?

**Prob. 4.** Explain by means of sketches why it is impossible for sustained oscillation to take place in the plate circuit of the amplifier in Fig. 164 when the natural period of the plate circuit is equal to twice the period of the applied pulses.

**Prob. 5.** Is sustained oscillation possible in the tuned amplifier in Fig. 164 when the natural period of the plate circuit is  $\frac{1}{3}$  the period of the applied pulses? Explain.

**Prob. 6.** In Fig. 169, the rectangular-pulse generator voltage is 50 volts.  $T = 10$  microseconds, the d-c generator voltage is 300 volts,  $r = 120$  ohms, and  $R = 2,800$  ohms. The filter in Fig. 170a is used with  $R_L = 80$  ohms and a filtering ratio of 6.3 per cent is achieved. What is the value of  $C$ ?

**Prob. 7.** In Prob. 6 suppose  $R_L = 0$ . What value of  $C$  is then required to achieve a filtering ratio of 6.3 per cent?

**Prob. 8.** The plate load resistor  $R_1$  in the pulse amplifier shown in Fig. 176 is changed from  $10^4$  ohms to  $5 \times 10^3$  ohms. All other values are the same as listed on page 252.

- What is the maximum pulse gain of the amplifier?
- What is the minimum pulse width that can be amplified without encountering a decrease in gain?



# APPENDIX I

TABLE V.—EXPONENTIALS [ $e^x$  AND  $e^{-x}$ ]

$x$	$e^x$	Diff.	$x$	$e^x$	Diff.	$x$	$e^x$	$x$	$e^{-x}$	Diff.	$x$	$e^{-x}$	$x$	$e^{-x}$
0.00	1.000	10	0.50	1.649	16	1.0	2.718*	0.00	1.000	-10	0.50	.607	1.0	.368*
.01	1.010	10	.51	1.665	17	.1	3.004	.01	0.990	-10	.51	.600	.1	.333
.02	1.020	10	.52	1.682	17	.2	3.320	.02	.980	-10	.52	.595	.2	.301
.03	1.030	10	.53	1.699	17	.3	3.669	.03	.970	-10	.53	.589	.3	.273
.04	1.041	11	.54	1.716	17	.4	4.055	.04	.961	-9	.54	.583	.4	.247
		10								-10				
0.05	1.051	11	0.55	1.733	18	1.5	4.482	0.05	.951	-9	0.55	.577	1.5	.223
.06	1.062	11	.56	1.751	17	.6	4.953	.06	.942	-10	.56	.571	.6	.202
.07	1.073	10	.57	1.768	18	.7	5.474	.07	.932	-9	.57	.566	.7	.183
.08	1.083	10	.58	1.786	18	.8	6.050	.08	.923	-9	.58	.560	.8	.165
.09	1.094	11	.59	1.804	18	.9	6.686	.09	.914	-9	.59	.554	.9	.150
		11								-9				
0.10	1.105	11	0.60	1.822	18	2.0	7.389	0.10	.905	-9	0.60	.549	2.0	.135
.11	1.116	11	.61	1.840	19	.1	8.166	.11	.896	-9	.61	.543	.1	.122
.12	1.127	11	.62	1.859	19	.2	9.025	.12	.887	-9	.62	.538	.2	.111
.13	1.139	12	.63	1.878	19	.3	9.974	.13	.878	-9	.63	.533	.3	.100
.14	1.150	12	.64	1.896	20	.4	11.02	.14	.869	-8	.64	.527	.4	.0907
		12								-8				
0.15	1.162	12	0.65	1.916	19	2.5	12.18	0.15	.861	-9	0.65	.522	2.5	.0821
.16	1.174	11	.66	1.935	19	.6	13.46	.16	.852	-8	.66	.517	.6	.0743
.17	1.185	12	.67	1.954	20	.7	14.88	.17	.844	-9	.67	.512	.7	.0672
.18	1.197	12	.68	1.974	20	.8	16.44	.18	.835	-8	.68	.507	.8	.0608
.19	1.209	12	.69	1.994	20	.9	18.17	.19	.827	-8	.69	.502	.9	.0550
		12								-8				
0.20	1.221	13	0.70	2.014	20	3.0	20.09	0.20	.819	-8	0.70	.497	3.0	.0498
.21	1.234	12	.71	2.034	20	.1	22.20	.21	.811	-8	.71	.492	.1	.0450
.22	1.246	13	.72	2.054	21	.2	24.53	.22	.803	-8	.72	.487	.2	.0408
.23	1.259	12	.73	2.075	21	.3	27.11	.23	.795	-8	.73	.482	.3	.0369
.24	1.271	13	.74	2.096	21	.4	29.96	.24	.787	-8	.74	.477	.4	.0334
		13								-8				
0.25	1.284	13	0.75	2.117	21	3.5	33.12	0.25	.779	-8	0.75	.472	3.5	.0302
.26	1.297	13	.76	2.138	22	.6	36.60	.26	.771	-8	.76	.468	.6	.0273
.27	1.310	13	.77	2.160	21	.7	40.45	.27	.763	-7	.77	.463	.7	.0247
.28	1.323	13	.78	2.181	21	.8	44.70	.28	.756	-8	.78	.458	.8	.0224
.29	1.336	14	.79	2.203	22	.9	49.40	.29	.748	-7	.79	.454	.9	.0202
		14								-7				
0.30	1.350	13	0.80	2.226	22	4.0	54.60	0.30	.741	-8	0.80	.449	4.0	.0183
.31	1.363	14	.81	2.248	22	.1	60.34	.31	.733	-7	.81	.445	.1	.0166
.32	1.377	14	.82	2.270	23	.2	66.69	.32	.726	-7	.82	.440	.2	.0150
.33	1.391	14	.83	2.293	23	.3	73.70	.33	.719	-7	.83	.436	.3	.0136
.34	1.405	14	.84	2.316	24	.4	81.45	.34	.712	-7	.84	.432	.4	.0123
		14								-7				
0.35	1.419	14	0.85	2.340	23	4.5	90.02	0.35	.705	-7	0.85	.427	4.5	.0111
.36	1.433	15	.86	2.363	24	.6	98.98	.36	.698	-7	.86	.423	.6	.00974
.37	1.448	14	.87	2.387	24	5.0	148.4	.37	.691	-7	.87	.419	5.0	.00674
.38	1.462	15	.88	2.411	24	6.0	403.4	.38	.684	-7	.88	.415	6.0	.00248
.39	1.477	15	.89	2.435	25	7.0	1097.	.39	.677	-7	.89	.411	7.0	.000912
		15								-7				
0.40	1.492	15	0.90	2.460	24	8.0	2981.	0.40	.670	-6	0.90	.407	8.0	.000335
.41	1.507	15	.91	2.484	25	9.0	8103.	.41	.664	-7	.91	.403	9.0	.000123
.42	1.522	15	.92	2.509	26	10.0	22026.	.42	.657	-6	.92	.399	10.0	.000045
.43	1.537	15	.93	2.535	25	$\pi/2$	4.810	.43	.651	-7	.93	.395	$\pi/2$	.208
.44	1.553	15	.94	2.560	26	$2\pi/2$	23.14	.44	.644	-6	.94	.391	$2\pi/2$	.0432
		15								-6				
0.45	1.568	16	0.95	2.586	26	$3\pi/2$	111.3	0.45	.638	-7	0.95	.387	$3\pi/2$	.00898
.46	1.584	16	.96	2.612	26	$4\pi/2$	535.5	.46	.631	-6	.96	.383	$4\pi/2$	.00187
.47	1.600	16	.97	2.638	26	$5\pi/2$	2576.	.47	.625	-6	.97	.379	$5\pi/2$	.000388
.48	1.616	16	.98	2.664	26	$6\pi/2$	12392.	.48	.619	-6	.98	.375	$6\pi/2$	.000081
.49	1.632	17	.99	2.691	27	$7\pi/2$	59610.	.49	.613	-6	.99	.372	$7\pi/2$	.000017
		17								-6			$8\pi/2$	.000003
0.50	1.649		1.00	2.718		$8\pi/2$	286751.	0.50	0.607		1.00	.368		

\* Note: Do not interpolate in this column.

$$e = 2.71828 \quad 1/e = 0.367879 \quad \log_{10} e = 0.4343 \quad 1/(0.4343) = 2.3026$$

$$\log_{10} (0.4343) = 1.6378 \quad \log_{10} (e^x) = x(0.4343)$$

From Marks "Mechanical Engineers' Handbook."

TABLE VI.—HYPERBOLIC SINES [ $\sinh x = \frac{1}{2}(e^x - e^{-x})$ ]

$x$	0	1	2	3	4	5	6	7	8	9	Avg. diff.
<b>0.0</b>	.0000	.0100	.0200	.0300	.0400	.0500	.0600	.0701	.0801	.0901	100
1	.1002	.1102	.1203	.1304	.1405	.1506	.1607	.1708	.1810	.1911	101
2	.2013	.2115	.2218	.2320	.2423	.2526	.2629	.2733	.2837	.2941	103
3	.3045	.3150	.3255	.3360	.3466	.3572	.3678	.3785	.3892	.4000	106
4	.4108	.4216	.4325	.4434	.4543	.4653	.4764	.4875	.4986	.5098	110
<b>0.5</b>	.5211	.5324	.5438	.5552	.5666	.5782	.5897	.6014	.6131	.6248	116
6	.6367	.6485	.6605	.6725	.6846	.6967	.7090	.7213	.7336	.7461	122
7	.7586	.7712	.7838	.7966	.8094	.8223	.8353	.8484	.8615	.8748	130
8	.8881	.9015	.9150	.9286	.9423	.9561	.9700	.9840	.9981	1.012	138
9	1.027	1.041	1.055	1.070	1.085	1.099	1.114	1.129	1.145	1.160	15
<b>1.0</b>	1.175	1.191	1.206	1.222	1.238	1.254	1.270	1.286	1.303	1.319	16
1	1.336	1.352	1.369	1.386	1.403	1.421	1.438	1.456	1.474	1.491	17
2	1.509	1.528	1.546	1.564	1.583	1.602	1.621	1.640	1.659	1.679	19
3	1.698	1.718	1.738	1.758	1.779	1.799	1.820	1.841	1.862	1.883	21
4	1.904	1.926	1.948	1.970	1.992	2.014	2.037	2.060	2.083	2.106	22
<b>1.5</b>	2.129	2.153	2.177	2.201	2.225	2.250	2.274	2.299	2.324	2.350	25
6	2.376	2.401	2.428	2.454	2.481	2.507	2.535	2.562	2.590	2.617	27
7	2.646	2.674	2.703	2.732	2.761	2.790	2.820	2.850	2.881	2.911	30
8	2.942	2.973	3.005	3.037	3.069	3.101	3.134	3.167	3.200	3.234	33
9	3.268	3.303	3.337	3.372	3.408	3.443	3.479	3.516	3.552	3.589	36
<b>2.0</b>	3.627	3.665	3.703	3.741	3.780	3.820	3.859	3.899	3.940	3.981	39
1	4.022	4.064	4.106	4.148	4.191	4.234	4.278	4.322	4.367	4.412	44
2	4.457	4.503	4.549	4.596	4.643	4.691	4.739	4.788	4.837	4.887	48
3	4.937	4.988	5.039	5.090	5.142	5.195	5.248	5.302	5.356	5.411	53
4	5.466	5.522	5.578	5.635	5.693	5.751	5.810	5.869	5.929	5.989	58
<b>2.5</b>	6.050	6.112	6.174	6.237	6.300	6.365	6.429	6.495	6.561	6.627	64
6	6.695	6.763	6.831	6.901	6.971	7.042	7.113	7.185	7.258	7.332	71
7	7.406	7.481	7.557	7.634	7.711	7.789	7.868	7.948	8.028	8.110	79
8	8.192	8.275	8.359	8.443	8.529	8.615	8.702	8.790	8.879	8.969	87
9	9.060	9.151	9.244	9.337	9.431	9.527	9.623	9.720	9.819	9.918	96
<b>3.0</b>	10.02	10.12	10.22	10.32	10.43	10.53	10.64	10.75	10.86	10.97	11
1	11.08	11.19	11.30	11.42	11.53	11.65	11.76	11.88	12.00	12.12	12
2	12.25	12.37	12.49	12.62	12.75	12.88	13.01	13.14	13.27	13.40	13
3	13.54	13.67	13.81	13.95	14.09	14.23	14.38	14.52	14.67	14.82	14
4	14.97	15.12	15.27	15.42	15.58	15.73	15.89	16.05	16.21	16.38	16
<b>3.5</b>	16.54	16.71	16.88	17.05	17.22	17.39	17.57	17.74	17.92	18.10	17
6	18.29	18.47	18.66	18.84	19.03	19.22	19.42	19.61	19.81	20.01	19
7	20.21	20.41	20.62	20.83	21.04	21.25	21.46	21.68	21.90	22.12	21
8	22.34	22.56	22.77	23.02	23.25	23.49	23.72	23.96	24.20	24.45	24
9	24.69	24.94	25.19	25.44	25.70	25.96	26.22	26.48	26.75	27.02	26
<b>4.0</b>	27.29	27.56	27.84	28.12	28.40	28.69	28.98	29.27	29.56	29.86	29
1	30.16	30.47	30.77	31.08	31.39	31.71	32.03	32.35	32.68	33.00	32
2	33.34	33.67	34.01	34.35	34.70	35.05	35.40	35.75	36.11	36.48	35
3	36.84	37.21	37.59	37.97	38.35	38.73	39.12	39.52	39.91	40.31	39
4	40.72	41.13	41.54	41.96	42.38	42.81	43.24	43.67	44.11	44.56	43
<b>4.5</b>	45.00	45.46	45.91	46.37	46.84	47.31	47.79	48.27	48.75	49.24	47
6	49.74	50.24	50.74	51.25	51.77	52.29	52.81	53.34	53.88	54.42	52
7	54.97	55.52	56.08	56.64	57.21	57.79	58.37	58.96	59.55	60.15	58
8	60.75	61.36	61.98	62.60	63.23	63.87	64.51	65.16	65.81	67.47	64
9	67.14	67.82	68.50	69.19	69.88	70.58	71.29	72.01	72.73	73.46	71
<b>5.0</b>	74.20										

If  $x > 5$ ,  $\sinh x = \frac{1}{2}(e^x)$  and  $\log_{10} \sinh x = (0.4343)x + 0.6990 - 1$ , correct to four significant figures.

From Marks, "Mechanical Engineers' Handbook."

TABLE VII.—HYPERBOLIC COSINES [ $\cosh x = \frac{1}{2}(\epsilon^x + \epsilon^{-x})$ ]

$x$	0	1	2	3	4	5	6	7	8	9	Avg. diff.
<b>0.0</b>	1.000	1.000	1.000	1.000	1.001	1.001	1.002	1.002	1.003	1.004	1
1	1.005	1.006	1.007	1.008	1.010	1.011	1.013	1.014	1.016	1.018	2
2	1.020	1.022	1.024	1.027	1.029	1.031	1.034	1.037	1.039	1.042	3
3	1.045	1.048	1.052	1.055	1.058	1.062	1.066	1.069	1.073	1.077	4
4	1.081	1.085	1.090	1.094	1.098	1.103	1.108	1.112	1.117	1.122	5
<b>0.5</b>	1.128	1.133	1.138	1.144	1.149	1.155	1.161	1.167	1.173	1.179	6
6	1.185	1.192	1.198	1.205	1.212	1.219	1.226	1.233	1.240	1.248	7
7	1.255	1.263	1.271	1.278	1.287	1.295	1.303	1.311	1.320	1.329	8
8	1.337	1.346	1.355	1.365	1.374	1.384	1.393	1.403	1.413	1.423	10
9	1.433	1.443	1.454	1.465	1.475	1.486	1.497	1.509	1.520	1.531	11
<b>1.0</b>	1.543	1.555	1.567	1.579	1.591	1.604	1.616	1.629	1.642	1.655	13
1	1.669	1.682	1.696	1.709	1.723	1.737	1.752	1.766	1.781	1.796	14
2	1.811	1.826	1.841	1.857	1.872	1.888	1.905	1.921	1.937	1.954	16
3	1.971	1.988	2.005	2.023	2.040	2.058	2.076	2.095	2.113	2.132	18
4	2.151	2.170	2.189	2.209	2.229	2.249	2.269	2.290	2.310	2.331	20
<b>1.5</b>	2.352	2.374	2.395	2.417	2.439	2.462	2.484	2.507	2.530	2.554	23
6	2.577	2.601	2.625	2.650	2.675	2.700	2.725	2.750	2.776	2.802	25
7	2.828	2.855	2.882	2.909	2.936	2.964	2.992	3.021	3.049	3.078	28
8	3.107	3.137	3.167	3.197	3.228	3.259	3.290	3.321	3.353	3.385	31
9	3.418	3.451	3.484	3.517	3.551	3.585	3.620	3.655	3.690	3.726	34
<b>2.0</b>	3.762	3.799	3.835	3.873	3.910	3.948	3.987	4.026	4.065	4.104	38
1	4.144	4.185	4.226	4.267	4.309	4.351	4.393	4.436	4.480	4.524	42
2	4.568	4.613	4.658	4.704	4.750	4.797	4.844	4.891	4.939	4.988	47
3	5.037	5.087	5.137	5.188	5.239	5.290	5.343	5.395	5.449	5.503	52
4	5.557	5.612	5.667	5.723	5.780	5.837	5.895	5.954	6.013	6.072	58
<b>2.5</b>	6.132	6.193	6.255	6.317	6.379	6.443	6.507	6.571	6.636	6.702	64
6	6.769	6.836	6.904	6.973	7.042	7.112	7.183	7.255	7.327	7.400	70
7	7.473	7.548	7.623	7.699	7.776	7.853	7.932	8.011	8.091	8.171	78
8	8.253	8.335	8.418	8.502	8.587	8.673	8.759	8.847	8.935	9.024	86
9	9.115	9.206	9.298	9.391	9.484	9.579	9.675	9.772	9.869	9.968	95
<b>3.0</b>	10.07	10.17	10.27	10.37	10.48	10.58	10.69	10.79	10.90	11.01	11
1	11.12	11.23	11.35	11.46	11.57	11.69	11.81	11.92	12.04	12.16	12
2	12.29	12.41	12.53	12.66	12.79	12.91	13.04	13.17	13.31	13.44	13
3	13.57	13.71	13.85	13.99	14.13	14.27	14.41	14.56	14.70	14.85	14
4	15.00	15.15	15.30	15.45	15.61	15.77	15.92	16.08	16.25	16.41	16
<b>3.5</b>	16.57	16.74	16.91	17.08	17.25	17.42	17.60	17.77	17.95	18.13	17
6	18.31	18.50	18.68	18.87	19.06	19.25	19.44	19.64	19.84	20.03	19
7	20.24	20.44	20.64	20.85	21.06	21.27	21.49	21.70	21.92	22.14	21
8	22.36	22.59	22.81	23.04	23.27	23.51	23.74	23.98	24.22	24.47	23
9	24.71	24.96	25.21	25.46	25.72	25.98	26.24	26.50	26.77	27.04	26
<b>4.0</b>	27.31	27.58	27.86	28.14	28.42	28.71	29.00	29.29	29.58	29.88	29
1	30.18	30.48	30.79	31.10	31.41	31.72	32.04	32.37	32.69	33.02	32
2	33.35	33.69	34.02	34.37	34.71	35.06	35.41	35.77	36.13	36.49	35
3	36.86	37.23	37.60	37.98	38.36	38.75	39.13	39.53	39.93	40.33	39
4	40.73	41.14	41.55	41.97	42.39	42.82	43.25	43.68	44.12	44.57	43
<b>4.5</b>	45.01	45.47	45.92	46.38	46.85	47.32	47.80	48.28	48.76	49.25	47
6	49.75	50.25	50.75	51.26	51.78	52.30	52.82	53.35	53.89	54.43	52
7	54.98	55.53	56.09	56.65	57.22	57.80	58.38	58.96	59.56	60.15	58
8	60.76	61.37	61.99	62.61	63.24	63.87	64.52	65.16	65.82	66.48	64
9	67.15	67.82	68.50	69.19	69.89	70.59	71.30	72.02	72.74	73.47	71
<b>5.0</b>	74.21										

If  $x > 5$ ,  $\cosh x = \frac{1}{2}(\epsilon^x)$  and  $\log_{10} \cosh x = (0.4343)x + 0.6990 - 1$ , correct to four significant figures.

From Marks, "Mechanical Engineers' Handbook."

TABLE VII.—HYPERBOLIC TANGENTS [ $\tanh x = (\epsilon^x - \epsilon^{-x})/(\epsilon^x + \epsilon^{-x})$   
=  $\sinh x / \cosh x$ ]

$x$	0	1	2	3	4	5	6	7	8	9	Avg. diff.
0.0	.0000	.0100	.0200	.0300	.0400	.0500	.0599	.0690	.0798	.0898	100
1	.0997	.1096	.1194	.1293	.1391	.1489	.1587	.1684	.1781	.1878	98
2	.1974	.2070	.2165	.2260	.2355	.2449	.2543	.2636	.2729	.2821	94
3	.2913	.3004	.3095	.3185	.3275	.3364	.3452	.3540	.3627	.3714	89
4	.3800	.3885	.3969	.4053	.4137	.4219	.4301	.4382	.4462	.4542	82
0.5	.4621	.4700	.4777	.4854	.4930	.5005	.5080	.5154	.5227	.5299	75
6	.5370	.5441	.5511	.5581	.5649	.5717	.5784	.5850	.5915	.5980	67
7	.6044	.6107	.6169	.6231	.6291	.6352	.6411	.6469	.6527	.6584	60
8	.6640	.6696	.6751	.6805	.6858	.6911	.6963	.7014	.7064	.7114	52
9	.7163	.7211	.7259	.7306	.7352	.7398	.7443	.7487	.7531	.7574	45
1.0	.7616	.7658	.7699	.7739	.7779	.7818	.7857	.7895	.7932	.7969	39
1	.8005	.8044	.8076	.8110	.8144	.8178	.8210	.8243	.8275	.8306	33
2	.8337	.8367	.8397	.8426	.8455	.8483	.8511	.8538	.8565	.8591	28
3	.8617	.8643	.8668	.8693	.8717	.8741	.8764	.8787	.8810	.8832	24
4	.8854	.8875	.8896	.8917	.8937	.8957	.8977	.8996	.9015	.9033	20
1.5	.9052	.9069	.9087	.9104	.9121	.9138	.9154	.9170	.9186	.9202	17
6	.9217	.9232	.9246	.9261	.9275	.9289	.9302	.9316	.9329	.9342	14
7	.9354	.9367	.9379	.9391	.9402	.9414	.9425	.9436	.9447	.9458	11
8	.9468	.9478	.9488	.9498	.9508	.9518	.9527	.9536	.9545	.9554	9
9	.9562	.9571	.9579	.9587	.9595	.9603	.9611	.9619	.9626	.9633	8
2.0	.9640	.9647	.9654	.9661	.9668	.9674	.9680	.9687	.9693	.9699	6
1	.9705	.9710	.9716	.9722	.9727	.9732	.9738	.9743	.9748	.9753	5
2	.9757	.9762	.9767	.9771	.9776	.9780	.9785	.9789	.9793	.9797	4
3	.9801	.9805	.9809	.9812	.9816	.9820	.9823	.9827	.9830	.9834	4
4	.9837	.9840	.9843	.9846	.9849	.9852	.9855	.9858	.9861	.9863	3
2.5	.9866	.9869	.9871	.9874	.9876	.9879	.9881	.9884	.9886	.9888	2
6	.9890	.9892	.9895	.9897	.9899	.9901	.9903	.9905	.9906	.9908	2
7	.9910	.9912	.9914	.9915	.9917	.9919	.9920	.9922	.9923	.9925	2
8	.9926	.9928	.9929	.9931	.9932	.9933	.9935	.9936	.9937	.9938	1
2.9	.9940	.9941	.9942	.9943	.9944	.9945	.9946	.9947	.9949	.9950	1
3.	.9951	.9959	.9967	.9973	.9978	.9982	.9985	.9988	.9990	.9992	4
4.	.9993	.9995	.9996	.9996	.9997	.9998	.9998	.9998	.9999	.9999	1
5.	.9999	If $x > 5$ , $\tanh x = 1.0000$ to four decimal places.									

TABLE IX.—MULTIPLES OF 0.4343 ( $0.43429448 = \log_{10} e$ )

$x$	0	1	2	3	4	5	6	7	8	9
0.	0.0000	0.0434	0.0869	0.1303	0.1737	0.2171	0.2606	0.3040	0.3474	0.3909
1.	0.4343	0.4777	0.5212	0.5646	0.6080	0.6514	0.6949	0.7383	0.7817	0.8252
2.	0.8686	0.9120	0.9554	0.9989	1.0423	1.0857	1.1292	1.1726	1.2160	1.2595
3.	1.3029	1.3463	1.3897	1.4332	1.4766	1.5200	1.5635	1.6069	1.6503	1.6937
4.	1.7372	1.7806	1.8240	1.8675	1.9109	1.9543	1.9978	2.0412	2.0846	2.1280
5.	2.1715	2.2149	2.2583	2.3018	2.3452	2.3886	2.4320	2.4755	2.5189	2.5625
6.	2.6058	2.6492	2.6926	2.7361	2.7795	2.8229	2.8663	2.9098	2.9532	2.9966
7.	3.0401	3.0835	3.1269	3.1703	3.2138	3.2572	3.3006	3.3441	3.3875	3.4309
8.	3.4744	3.5178	3.5612	3.6046	3.6481	3.6915	3.7349	3.7784	3.8218	3.8652
9.	3.9087	3.9521	3.9955	4.0389	4.0824	4.1258	4.1692	4.2127	4.2561	4.2995

From Marks, "Mechanical Engineers' Handbook."

TABLE X.—MULTIPLES OF 2.3026 ( $2.3025851 = 1/0.4343$ )

$x$	0	1	2	3	4	5	6	7	8	9
0.	0.0000	0.2303	0.4605	0.6908	0.9210	1.1513	1.3816	1.6118	1.8421	2.0723
1.	2.3026	2.5328	2.7631	2.9934	3.2236	3.4539	3.6841	3.9144	4.1447	4.3749
2.	4.6052	4.8354	5.0657	5.2959	5.5262	5.7565	5.9867	6.2170	6.4472	6.6775
3.	6.9078	7.1380	7.3683	7.5985	7.8288	8.0590	8.2893	8.5196	8.7498	8.9801
4.	9.2103	9.4406	9.6709	9.9011	10.131	10.362	10.592	10.822	11.052	11.283
5.	11.513	11.743	11.973	12.204	12.434	12.664	12.894	13.125	13.355	13.585
6.	13.816	14.046	14.276	14.506	14.737	14.967	15.197	15.427	15.658	15.888
7.	16.118	16.348	16.579	16.809	17.039	17.269	17.500	17.730	17.960	18.190
8.	18.421	18.651	18.881	19.111	19.342	19.572	19.802	20.032	20.263	20.493
9.	20.723	20.954	21.184	21.414	21.644	21.875	22.105	22.335	22.565	22.796

From Marks, "Mechanical Engineers' Handbook."

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